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Algebraicity of solutions of functional equations with one catalytic variable[1]

Algebraic and Enumerative Combinatorics (Waterloo, Canada)

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[1] Joint work with Hadrien Notarantonio

Motivating examples

- Let $F(t, u) = \sum_{n,k \geq 0} a_{n,k} t^n u^k$ be the generating function of walks in $\mathbb{N}^2$ which have n steps in $\{\nearrow, \searrow\}$ and end at level (height) k. One finds:

$$F(t, u) = 1 + tuF(t, u) + t\frac{F(t, u) - F(t, 0)}{u}.$$

  It follows that: $\quad F(t, 0) = \dfrac{1 - \sqrt{1 - 4t}}{2t}$.   In particular: $F(t, 0) \in \overline{\mathbb{Q}(t)}$.

- Modelling [Bonichon, Bousquet-Mélou, Dorbec, Pennarun, 2006] special Eulerian planar orientations gives rise to:

$$\begin{cases} F_1(t, u) = 1 + t \cdot \left(u + 2uF_1(t, u)^2 + 2uF_2(t, 1) + u\frac{F_1(t,u) - uF_1(t,1)}{u-1} \right), \\ F_2(t, u) = t \cdot \left(2uF_1(t, u)F_2(t, u) + uF_1(t, u) + uF_2(t, 1) + u\frac{F_2(t,u) - uF_2(t,1)}{u-1} \right). \end{cases}$$

  Again: $G = F_1(t, 0)$ and $F_2(t, 0)$ are algebraic functions, for example:

$$64t^3 G^3 + 2t\left(24t^2 - 36t + 1\right) G^2 - \left(15t^3 - 9t^2 - 19t + 1\right) G + t^3 + 27t^2 - 19t + 1 = 0.$$

Discrete Differential Equations (DDEs) with one catalytic variable

- The divided difference operator (discrete derivative):

$$\Delta_a : \mathbb{Q}[u][[t]] \rightarrow \mathbb{Q}[u][[t]],$$
$$F(t, u) \mapsto \frac{F(t, u) - F(t, a)}{u - a}.$$

- Δ_a^j is the j-th iteration of Δ. Explicitly:

$$\Delta_a^{j+1} F(t, u) = \frac{F(t, u) - F(t, a) - (u - a)\partial_u F(t, a) - \cdots - \frac{(u-a)^j}{j!}\partial_u^j F(t, a)}{(u - a)^{j+1}}$$

- For polynomials $f(u) \in \mathbb{Q}[u]$ and $Q \in \mathbb{Q}[x, y_1 \ldots, y_k, t, u]$ consider the equation

$$F(t, u) = f(u) + t \cdot Q(F(t, u), \Delta_a F(t, u), \ldots, \Delta_a^k F(t, u), t, u), \qquad \textbf{(DDE)}$$

  where $a \in \mathbb{Q}$ (usually 0 or 1) and $k \in \mathbb{N}$ (the order of the **DDE**).

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Some history

- In 1960s, Tutte reduced many combinatorial problems to studying **DDEs**.
  E.g.: [Tutte, 1962] and [Brown, Tutte 1964]
- In 1986 Popescu proved the "General Néron desingularization" [Popescu, 1986].
- [Bandarier, Flajolet 2002]: Universal "Kernel method" for linear **DDEs**.
- In 2006: The unique solution of any **DDE** is an algebraic function and effective
  method to compute the minimal polynomial [Bousquet-Mélou, Jehanne, 2006].
- In 2015 [Hauser, Rond] organize a conference on "Artin Approximation":
  Popescu's Theorem implies also algebraicity of **DDEs** (but in a non-effective way).
- [Buchacher, Kauers, 2020]: Linear systems of **DDEs** have algebraic solutions
  (effective proof).
- New: [Notarantonio, Y., 2022]: Effective proof that general systems of **DDEs**
  have algebraic solutions.

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Main Result for scalar equations

Theorem (Bousquet-Mélou, Jehanne, 2006)

Let $k \geq 1, a \in \mathbb{Q}$ and $Q \in \mathbb{Q}[x, y_1, \ldots, y_k, t, u]$, $f(u) \in \mathbb{Q}[u]$. There exists a unique solution $F(t, u) \in \mathbb{Q}[u][\![t]\!]$ of the functional equation

$$F(t, u) = f(u) + t \cdot Q(F(t, u), \Delta_a F(t, u), \ldots, \Delta_a^k F(t, u), t, u),$$

and $F(t, u)$ is algebraic over $\mathbb{Q}(t, u)$. Moreover, there exists an algorithm for computing the minimal polynomial of $F(t, u)$.

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Sketch of proof (generic case)

- Let E be the polynomial in $x, z_0, \ldots, z_{k-1}, t, u$ that is induced by **DDE**:

$$E(u) := E(\underbrace{F(t, u)}_{x}, \underbrace{F(t, 0)}_{z_0}, \ldots, \underbrace{\partial_u^{k-1} F(t, 0)}_{z_{k-1}}, t, u) = 0.$$

- Take the derivative of $E(u) = 0$ with respect to u:

$$\partial_u F(t, u) \cdot \partial_x E(u) + \partial_u E(u) = 0. \qquad (\partial_x E(u) \text{ is the "kernel")}$$

- Any solution $u = U(t)$ of $\partial_x E(u) = 0$ also implies $\partial_u E(u) = 0$.
- Obtain 3 equations ($E(u) = 0, \partial_x E(u) = 0, \partial_u E(u) = 0$) for $k + 1 + 1$ variables.
- If $\partial_x E(u) = 0$ has k distinct solutions $U_1, \ldots, U_k$, we can consider:

$$\mathcal{S}_{\text{dup}} := \begin{cases} E(u_i) = 0, \\ \partial_x E(u_i) = 0, \qquad \text{for } i = 1, \ldots, k. \\ \partial_u E(u_i) = 0. \end{cases} \qquad \begin{cases} F(t, U_i) \leftrightarrow x_i, \\ \partial_u^i F(t, 0) \leftrightarrow z_i, \\ U_i \leftrightarrow u_i. \end{cases}$$

- We find: $3k$ equations and $3k$ variables. Hope: the system $\mathcal{S}_{\text{dup}}$ is 0-dimensional.
- In that case we can use elimination algorithms: find the annihilating polynomial.

Sketch of proof

- Two issues:
  1. $\partial_x E(u) = 0$ does not always have k distinct solutions.
  2. Is $\mathcal{S}_{\mathrm{dup}}$ really 0-dimensional?
- Given the **DDE**

$$F(t, u) = f(u) + t \cdot Q(F(t, u), \Delta_a F(t, u), \ldots, \Delta_a^k F(t, u), t, u),$$

  consider the perturbation **DDE$_\epsilon$**:

$$G(t, u, \epsilon) = f(u) + t^2 \cdot Q(G(t, u, \epsilon), \Delta_a G(t, u, \epsilon), \ldots, \Delta_a^k G(t, u, \epsilon), t^2, u) + t\epsilon^k \Delta^k G(t, u, \epsilon).$$

- $G(t, u, \epsilon)$ algebraic over $\mathbb{Q}(t, u, \epsilon) \Rightarrow F(t, u)$ algebraic over $\mathbb{Q}(t, u)$.

Lemma 1

Let E_G be the numerator of DDE$_\epsilon$. $\partial_x E_G(u) = 0$ has k distinct solutions in $\overline{\mathbb{Q}(\epsilon)}[\![t^{\frac{1}{\star}}]\!]$.

Lemma 2

The ideal $\langle \mathcal{S}_{\mathrm{dup}} \rangle : \det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}})^\infty$ is 0-dimensional and contains the minimal poly. of G.

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Sketch of proof (Lemma 1)

Lemma 1

Let E_G be the numerator of DDE_ϵ. $\partial_x E_G(u) = 0$ has k distinct solutions in $\overline{\mathbb{Q}(\epsilon)}[\![t^{\frac{1}{\star}}]\!]$.

- The equation $\partial_x E_G = 0$ has the form:

$$u^k = \epsilon^k t + t^2 \sum_{i=0}^{k} u^{k-i} \partial_{y_i} Q(F(t,u), \ldots, \Delta^k F(t,u), t^2, u)$$

- Newton-Raphson algorithm implies that we find k solutions of the form

$$U_\ell(t) = \epsilon \cdot t^{\frac{1}{k}} \cdot \zeta^\ell + O(t^{\frac{2}{k}}),$$

for ζ a primitive k-th root of unity.

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Sketch of proof (Lemma 2)

Lemma 2

The ideal $\langle \mathcal{S}_{\mathrm{dup}} \rangle : \det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}})^{\infty}$ is 0-dimensional and contains the minimal poly. of G.

- Recall: $E_G(u) := E_G(F(t, u), F(t, 0), \ldots, \partial_u^{k-1} F(t, 0), t, u)$ where E_G describes

$$G(t, u, \epsilon) = f(u) + t^2 \cdot Q(G(t, u, \epsilon), \Delta_a G(t, u, \epsilon), \ldots, \Delta_a^k G(t, u, \epsilon), t^2, u) + t\epsilon^k \Delta^k G(t, u, \epsilon).$$

$$\mathcal{S}_{\mathrm{dup}} := \left\{ E_G(u_i) = 0, \partial_x E_G(u_i) = 0, \partial_u E_G(u_i) = 0 : i = 1, \ldots, k \right\}$$

- Application of Hilbert's Nullstellensatz: Lemma 2 holds if the Jacobian matrix $\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}} \in \overline{\mathbb{Q}(\epsilon)}[\![t^{\frac{1}{\star}}]\!]^{3k \times 3k}$ of $\mathcal{S}_{\mathrm{dup}}$ is invertible at

$$(x_1, \ldots, x_k, u_1, \ldots, u_k, z_0, \ldots, z_{k-1}) =$$

$$(G(t, U_1), \ldots, G(t, U_k), U_1(t), \ldots, U_k(t), G(t, 0), \ldots, \partial_u^{k-1} G(t, 0)) \in \overline{\mathbb{Q}(\epsilon)}[\![t^{\frac{1}{\star}}]\!]^{3k}.$$

- Bousquet-Mélou and Jehanne compute the determinant explicitly:

$$\det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}}) = r \cdot \prod_{i<j}(\zeta^i - \zeta^j) \cdot \prod_{j=0}^{k} \left(\partial_x^2 E(U_j) \partial_u^2 E(U_j) - \partial_x \partial_u E(U_j)^2 \right) + O(t^k) \neq 0.$$

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Summary of the proof and algorithm in the scalar case

- Strategy of the proof:

  1 Given a **DDE** for $F(t, u)$ consider the perturbation **DDE**$_\epsilon$:

  $$G(t, u, \epsilon) = f(u) + t^2 \cdot Q(G(t, u, \epsilon), \Delta_a G(t, u, \epsilon), \ldots, \Delta_a^k G(t, u, \epsilon), t^2, u) + t\epsilon^k \Delta^k G(t, u, \epsilon).$$

  and let E_G be the defining polynomial in the the variables $x, z_0, \ldots, z_{k-1}, t, u, \epsilon$.

  2 Prove that $\partial_x E_G(u) = 0$ has k disctinct solutions $u = U(t)$ in $\overline{\mathbb{Q}(\epsilon)}[\![t^{\frac{1}{\star}}]\!]$.

  3 Define $\mathcal{S} = (E, \partial_x E_G, \partial_u E_G)$ and let $\mathcal{S}_{\mathrm{dup}}$ be the duplicated system.

  4 Show that $\langle \mathcal{S}_{\mathrm{dup}} \rangle : \det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}})^\infty$ is 0-dimensional by proving: $\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}}$ is invertible.

- Algorithm:

  1 Define $E_G \in \mathbb{Q}[x, z_0, \ldots, z_{k-1}, t, u, \epsilon]$ as the numerator of of **DDE**$_\epsilon$.

  2 Compute $\partial_x E$ and $\partial_u E$. Define $\mathcal{S}_{\mathrm{dup}}$ in $x_1, \ldots, x_n, u_1, \ldots, u_n, z_0, \ldots, z_{k-1}$.

  3 Saturate $\mathcal{S}_{\mathrm{dup}}$ by adding the equation $m \cdot \det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}}) - 1 = 0$ for a variable m.

  4 Compute a non-zero element of $\mathcal{S}_{\mathrm{sat}} \cap \mathbb{Q}[z_0, t]$.

New result: extension for systems

Theorem (Notarantonio, Y., 2022)

Let $n, k \geq 1$ be integers and $f_1, \ldots, f_n \in \mathbb{Q}[u]$, $Q_1, \ldots, Q_n \in \mathbb{Q}[y_1, \ldots, y_{n(k+1)}, t, u]$ be polynomials. Set $\nabla^k F := F, \Delta_a F, \ldots, \Delta_a^k F$. Then the system of equations

$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \ldots, \nabla^k F_n, t, u), \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \ldots, \nabla^k F_n, t, u) \end{cases}$$

admits a unique vector of solutions $(F_1, \ldots, F_n) \in \mathbb{Q}[u][[t]]^n$, and all its components are algebraic functions over $\mathbb{Q}(t, u)$.

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Example

Theorem (Notarantonio, Y., 2022)

A system DDEs with one catalytic variable admits a unique vector of solutions $(F_1, \ldots, F_n) \in \mathbb{Q}[u][[t]]^n$, and all its components are algebraic functions over $\mathbb{Q}(t, u)$.

Example (introduced and solved in [Bonichon,Bousquet-Mélou,Dorbec,Pennarun, 2006]):

$$\begin{cases} F_1(t, u) = 1 + t \cdot \left(u + 2uF_1(t, u)^2 + 2uF_2(t, 1) + u\frac{F_1(t,u) - uF_1(t,1)}{u-1} \right), \\ F_2(t, u) = t \cdot \left(2uF_1(t, u)F_2(t, u) + uF_1(t, u) + uF_2(t, 1) + u\frac{F_2(t,u) - uF_2(t,1)}{u-1} \right). \end{cases}$$

It holds: $G = F_1(t, 0)$ and $F_2(t, 0)$ are algebraic functions. For example:

$$64t^3G^3 + 2t\left(24t^2 - 36t + 1\right)G^2 - \left(15t^3 - 9t^2 - 19t + 1\right)G + t^3 + 27t^2 - 19t + 1 = 0.$$

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Generic system case

- The system
$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \ldots, \nabla^k F_n, t, u), \\ \vdots \qquad\qquad \vdots \\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \ldots, \nabla^k F_n, t, u) \end{cases}$$

  defines n polynomial equations $E_1 = 0, \ldots, E_n = 0$ in $\mathbb{Q}[x_1, \ldots, x_n, z_0, \ldots, z_{nk-1}, u, t]$.

- The "derivative" of $(E_1, \ldots, E_n)$ with respect to u:
$$\begin{pmatrix} \partial_{x_1} E_1 & \ldots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \ldots & \partial_{x_n} E_n \end{pmatrix} \cdot \begin{pmatrix} \partial_u G_1 \\ \vdots \\ \partial_u G_n \end{pmatrix} + \begin{pmatrix} \partial_u E_1 \\ \vdots \\ \partial_u E_n \end{pmatrix} = 0. \tag{1}$$

- Now define:
$$\text{Det} := \det \begin{pmatrix} \partial_{x_1} E_1 & \ldots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \ldots & \partial_{x_n} E_n \end{pmatrix} \quad \text{and} \quad \text{P} := \det \begin{pmatrix} \partial_{x_1} E_1 & \ldots & \partial_{x_1} E_{n-1} & \partial_{x_1} E_n \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_{n-1}} E_1 & \ldots & \partial_{x_{n-1}} E_{n-1} & \partial_{x_{n-1}} E_n \\ \partial_u E_1 & \ldots & \partial_u E_{n-1} & \partial_u E_n \end{pmatrix}.$$

- Observation: (1) and Det $= 0$ imply P $= 0$.
  Therefore we have: $n + 2$ **equations**, and $n + nk + 1$ **variables**.

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Generic system case

- Equations: $\mathcal{S} = \{E_1(u), \ldots, E_n(u), \mathrm{Det}(u), P(u)\}$.
- If $\mathrm{Det}(u) = 0$ has nk distinct solutions in $\overline{\mathbb{Q}}[\![t^{\frac{1}{\star}}]\!]$, then define the system:

$$\mathcal{S}_{\mathrm{dup}} \coloneqq \begin{cases} E_1(u_i) = E_2(u_i) = \cdots = E_n(u_i) = 0, \\ \mathrm{Det}(u_i) = 0, \\ P = 0. \end{cases} \quad \text{for } i = 1, \ldots, k.$$

- In total $(nk + 2n) \cdot k = nk(n+2)$ equations.
- Variables: $\underbrace{x_1, \ldots, x_{n^2 k}}_{F_i(U_i)}, \underbrace{z_0, \ldots, z_{nk-1}}_{\partial^j F_i(t,a)}, \underbrace{u_1, \ldots, u_{nk}}_{U_i} \Rightarrow n^2 k + nk + nk = nk(n+2)$.
- $\Rightarrow$ Can hope for a 0-dimensional system.

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Outline of the proof for systems in the general case

1. Given a system of **DDE**s for $F_i(t, u)$ consider the perturbed system **DDE**$_\epsilon$.

2. Prove that $\mathrm{Det}(u) = 0$ has *nk* disctinct solutions $u = U(t)$ in $\overline{\mathbb{Q}(\epsilon)}[\![t^{\frac{1}{\ast}}]\!]$.

3. Define $\mathcal{S} = (E_1, \ldots, E_n, \mathrm{Det}, P)$ and let $\mathcal{S}_{\mathrm{dup}}$ be the duplicated system in $nk(n + 2)$ variables.

4. Show: $\langle \mathcal{S}_{\mathrm{dup}} \rangle : \det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}})^\infty$ is 0-dimensional by proving: $\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}}$ is invertible.

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Sketch of proof: Step 1, deformation

$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \ldots, \nabla^k F_n, t, u), \\ \vdots \qquad\qquad\qquad \vdots \\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \ldots, \nabla^k F_n, t, u) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} G_1 = f_1(u) + t^{\alpha} \cdot Q_1(\nabla^k G_1, \nabla^k G_2, \ldots, \nabla^k G_n, t^{\alpha}, u) + t \cdot \epsilon^k \cdot \sum_{i=1}^{n} \gamma_{1,i} \cdot \Delta^k G_i, \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ G_n = f_n(u) + t^{\alpha} \cdot Q_n(\nabla^k G_1, \nabla^k G_2, \ldots, \nabla^k G_n, t^{\alpha}, u) + t \cdot \epsilon^k \cdot \sum_{i=1}^{n} \gamma_{n,i} \cdot \Delta^k G_i, \end{cases}$$

where $\gamma_{i,i} = i^k$ and $\gamma_{i,j} = t^{\beta}$, and $\alpha \gg \beta \gg 0$ are chosen sufficiently large.

- It holds: $G_i(t, u, \epsilon)$ algebraic over $\mathbb{Q}(t, u, \epsilon) \Rightarrow F_i(t, u)$ algebraic over $\mathbb{Q}(t, u)$.
- Each equation for G_i induces a polynomial equation:

$$E_i(\nabla^k G_1, \nabla^k G_2, \ldots, \nabla^k G_n, t, u, \epsilon) = 0, \quad i = 1, \ldots, n.$$

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Step 2: Definition of Det(u) and proof of nk distinct solutions

$$\text{Det} = \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_n} E_n \end{pmatrix}$$

Lemma 1

The equation $\text{Det}(u) = 0$ has nk distinct solutions in $\overline{\mathbb{Q}(\epsilon)}[\![t^{\frac{1}{\star}}]\!]$

- We have

$$\text{Det}(u) = \det \begin{pmatrix} -u^{m_1} + t\epsilon^k \gamma_{1,1} u^{m_1-k} & \cdots & t\epsilon^k \gamma_{1,n} u^{m_1-k} \\ \vdots & \ddots & \vdots \\ t\epsilon^k \gamma_{n,1} u^{m_n-k} & \cdots & -u^{m_n} + t\epsilon^k \gamma_{n,n} u^{m_n-k} \end{pmatrix} + O(t^{\alpha} u^{M-nk}).$$

- Using $\gamma_{i,i} = i^k$ and $\gamma_{i,j} = t^{\beta}$, we get $\text{Det} = \prod_{j=1}^{n}(-u^k + t\epsilon^k j^k) \bmod t^{n+1}$.
- Newton-Raphson algorithm:

$$U_i = \zeta^{\ell} \cdot j \cdot t^{\frac{1}{k}} + O(t^{\frac{2}{k}}), \quad \text{for } \ell = 1, \dots, k \text{ and } j = 1, \dots, n, \text{ with } \zeta^k = 1.$$

Outline of the proof for systems in the general case

1. Given a system of **DDE**s for $F_i(t, u)$ consider the perturbed system $\textbf{DDE}_\epsilon$.

2. Prove that $\text{Det}(u) = 0$ has *nk* disctinct solutions $u = U(t)$ in $\overline{\mathbb{Q}(\epsilon)}[\![t^{\frac{1}{*}}]\!]$.

3. Define $\mathcal{S} = (E_1, \ldots, E_n, \text{Det}, \text{P})$ and let $\mathcal{S}_{\text{dup}}$ be the duplicated system in $nk(n + 2)$ variables.

4. Show: $\langle \mathcal{S}_{\text{dup}} \rangle : \det(\text{Jac}_{\mathcal{S}_{\text{dup}}})^\infty$ is 0-dimensional by proving: $\text{Jac}_{\mathcal{S}_{\text{dup}}}$ is invertible.

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Step 4: $\mathsf{Jac}_{(\mathcal{S}_{\mathsf{dup}})}$ is invertible (very brief sketch)

$$\mathsf{Jac}_{(\mathcal{S}_{\mathsf{dup}})} = \begin{pmatrix} A_1 & & 0 & B_1 \\ & \ddots & & \vdots \\ 0 & & A_{nk} & B_{nk} \end{pmatrix} \in \overline{\mathbb{L}}[[t^{\frac{1}{\star}}]]^{nk(n+2) \times nk(n+2)},$$

$$A_i := \begin{pmatrix} \partial_{x_1} E_1^{(i)}(U_i) & \cdots & \partial_{x_n} E_1^{(i)}(U_i) & \partial_{u_i} E_1^{(i)}(U_i) \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} E_n^{(i)}(U_i) & \cdots & \partial_{x_n} E_n^{(i)}(U_i) & \partial_{u_i} E_n^{(i)}(U_i) \\ \partial_{x_1} \mathsf{Det}^{(i)}(U_i) & \cdots & \partial_{x_n} \mathsf{Det}^{(i)}(U_i) & \partial_{u_i} \mathsf{Det}^{(i)}(U_i) \\ \partial_{x_1} P^{(i)}(U_i) & \cdots & \partial_{x_n} P^{(i)}(U_i) & \partial_{u_i} P^{(i)}(U_i) \end{pmatrix}, B_i := \begin{pmatrix} \partial_{z_0} E_1^{(i)}(U_i) & \cdots & \partial_{z_{nk-1}} E_1^{(i)}(U_i) \\ \vdots & \ddots & \vdots \\ \partial_{z_0} E_n^{(i)}(U_i) & \cdots & \partial_{z_{nk-1}} E_n^{(i)}(U_i) \\ \partial_{z_0} \mathsf{Det}^{(i)}(U_i) & \cdots & \partial_{z_{nk-1}} \mathsf{Det}^{(i)}(U_i) \\ \partial_{z_0} P^{(i)}(U_i) & \cdots & \partial_{z_{nk-1}} P^{(i)}(U_i) \end{pmatrix}.$$

$$\Rightarrow \det(\mathsf{Jac}_{(\mathcal{S}_{\mathsf{dup}})}) = \pm \big(\prod_{i=1}^{nk} \det(\mathsf{Jac}_i(U_i)) \big) \cdot \det(\Lambda), \quad \text{for}$$

$$\mathsf{Jac}_i(u) \in \mathbb{Q}(\epsilon)[u][[t]]^{(n+1) \times (n+1)} \text{ and } \Lambda \in \overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{\star}}]]^{nk \times nk}$$

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Special Eulerian planar orientations

- [Bonichon, Bousquet-Mélou, Dorbec, Pennarun, 2006] consider and solve:

$$\begin{cases} F_1(t, u) = 1 + t \cdot \left(u + 2uF_1(t, u)^2 + 2uF_2(t, 1) + u\frac{F_1(t,u) - uF_1(t,1)}{u-1} \right), \\ F_2(t, u) = t \cdot \left(2uF_1(t, u)F_2(t, u) + uF_1(t, u) + uF_2(t, 1) + u\frac{F_2(t,u) - uF_2(t,1)}{u-1} \right). \end{cases}$$

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Special Eulerian planar orientations

- We get polynomial equations

$$\begin{cases} E_1 = (1 - x_1) \cdot (u - 1) + t \cdot (2u^2x_1^2 - u^2z_0 + 2u^2z_1 - 2ux_1^2 + u^2 + ux_1 - 2uz_1 - u), \\ E_2 = x_2 \cdot (1 - u) + t \cdot (2u^2x_1x_2 + u^2x_1 - 2ux_1x_2 - ux_1 + ux_2 - uz_1). \end{cases}$$

- Then define

$$\begin{cases} \mathrm{Det} = (4tu^2x_1 - 4tux_1 + tu - u + 1)(2tu^2x_1 - 2tux_1 + tu - u + 1), \\ \mathrm{P} = -2tx_1x_2 - tx_1 + tx_2 - tz_1 - x_2 + P_1 \cdot u + P_2 \cdot u^2 + P_3 \cdot u^3, \end{cases}$$

- $\mathcal{S}_{\mathrm{dup}} =$
  $(E_1(x_1, x_2, z_0, z_1, u_1), E_2(x_1, x_2, z_0, z_1, u_1), \mathrm{Det}(x_1, x_2, z_0, z_1, u_1), \mathrm{P}(x_1, x_2, z_0, z_1, u_1),$
  $(E_1(x_3, x_4, z_0, z_1, u_2), E_2(x_3, x_4, z_0, z_1, u_2), \mathrm{Det}(x_3, x_4, z_0, z_1, u_2), \mathrm{P}(x_3, x_4, z_0, z_1, u_2))$
- Compute a generator of $\langle \mathcal{S}_{\mathrm{dup}}, m \cdot (u_1 - u_2) - 1 \rangle \cap \mathbb{Q}[z_0, t]$.

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Special Eulerian planar orientations in Maple

```
E1 := numer(-x1+1 + 2*t*u*x1^2 + 2*t*u*z1 + t*u*(-u*z0+x1)/(u-1)+t*u);
E2 := numer(-x2 + 2*t*u*x1*x2 + t*u*x1 + t*u*z1 + t*u*(-u*z1+x2)/(u-1));
Jac := Matrix([[diff(E1, x1),diff(E1, x2)],[diff(E2, x1),diff(E2, x2)]]);
Det := LinearAlgebra[Determinant](Jac);
Pm := Matrix([[diff(E1, x1), diff(E2, x1)], [diff(E1, u), diff(E2, u)]]);
P := LinearAlgebra[Determinant](Pm);
S := [E1, E2, det, P];
S1 := op(subs(x1=x1,x1=x1,u=u1,S));
S2 := op(subs(x1=x1,x1=x1,u=u2,S));
Sdup := [S1,S2, m*(u1 - u2) - 1];
G := polynomial_elimination(Sdup, z0, t, 2);
```

$$(z0 - 1)(2tz0 + t - 1)(64t^3z0^3 + 48t^3z0^2 - 15t^3z0 - 72t^2z0^2 + t^3 + 9t^2z0 + \cdots) \cdots$$

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Summary and conclusion

- **Systems of DDEs** with one catalytic variable have algebraic solutions.

- There exists an algorithm for finding minimal polynomials of such solutions.

- There is ongoing work on improving the efficiency and effective handling of more catalytic variables.