When is a hypergeometric function algebraic?[1]

Differential Seminar (IHP, Paris)

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A&R TECH

April 14, 2026

[1] Joint work (arxiv.org/abs/2308.12855) with Florian Fürnsinn

Definitions

Hypergeometric sequence:

$$u_{n+1} = \frac{A(n)}{B(n)} u_n,$$

where $A(t), B(t) \in \mathbb{Q}[t]$ are polynomials.

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Hypergeometric function:

$${}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{x^n}{n!},$$

where $(a)_n := a(a+1)\cdots(a+n-1)$ denotes the rising factorial.

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Hypergeometric functions are generating functions of hypergeometric sequences.

Examples

- The logarithm:

$$ {}_2F_1\left[\begin{matrix}1,1\\2\end{matrix};x\right] = -\frac{\log(1-x)}{x} = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \cdots \in \mathbb{Q}[\![x]\!] $$

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- Catalan numbers:

$$ C_n = \binom{2n}{n}\frac{1}{n+1} \in \mathbb{Z}, \quad \frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{(n+2)}, \quad \sum_{n\geq 0} C_n x^n = {}_2F_1\left[\begin{matrix}\frac{1}{2},1\\2\end{matrix};4x\right] $$

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- Chebychev numbers:

$$ T_n = \frac{(30n)!n!}{(15n)!(10n)!(6n)!} \in \mathbb{Z}, \quad \sum_{n\geq 0} T_n x^n = {}_8F_7\left[\begin{matrix}\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}\\\frac{1}{5},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{4}{5}\end{matrix};2^{14}\cdot 3^9\cdot 5^5\cdot x\right] $$

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- More algebraic series:

$$ {}_3F_2\left[\begin{matrix}1/2,\sqrt{2}+1,-\sqrt{2}+1\\\sqrt{2},-\sqrt{2}\end{matrix};4x\right] = \frac{(7x-1)(2x-1)}{(1-4x)^{5/2}} = 1+x-6x^2+\cdots \in \mathbb{Z}[\![x]\!] $$

Definitions

A series $f(x) \in \mathbb{Q}[[x]]$ is called algebraic if there exists $P(x, y) \in \mathbb{Q}[x, y] \setminus \{0\}$, such that $P(x, f(x)) = 0$.

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Theorem (Eisenstein 1852, Heine 1854)

Any algebraic $f(x) \in \mathbb{Q}[\![x]\!]$ is almost integral.

$\Rightarrow$ Arithmetic proof that $-\log(1-x)/x = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \cdots$ is transcendental.

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Conjecture (weak form of p-curvature conjecture, Grothendieck 1969, Bézivin 1991)

A linear ODE of order n has n linearly independent algebraic solutions if and only if it has n linearly independent almost integral solutions.

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Any hypergeometric function $F(x) \in \mathbb{Q}[[x]]$ is D-finite:

$$x(\theta + a_1) \cdots (\theta + a_p) F(x) = \theta(\theta + b_1 - 1) \cdots (\theta + b_{p-1} - 1) F(x), \qquad \left(\theta = x \frac{\mathrm{d}}{\mathrm{d}x}\right).$$

For example, $F(x) = {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; x\right]$ satisfies

$$x(1-x)F''(x) + [c - (a+b+1)x]\, F'(x) - ab\, F(x) = 0.$$

Main Question of this Talk

Which hypergeometric functions are algebraic?

$${}_2F_1\left[\begin{matrix}1,1\\2\end{matrix};x\right]$$

$${}_2F_1\left[\begin{matrix}\frac{1}{2},1\\2\end{matrix};4x\right]$$

$${}_3F_2\left[\begin{matrix}1/2,\sqrt{2}+1,-\sqrt{2}+1\\\sqrt{2},-\sqrt{2}\end{matrix};4x\right]$$

$${}_6F_5\left[\begin{matrix}\frac{1}{4},\frac{1}{2},\frac{3}{4},3,3,1\\\frac{1}{3},\frac{2}{3},4,2,2\end{matrix};\frac{256}{27}x\right]$$

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$${}_3F_2\left[\begin{matrix}\frac{5}{6},\frac{1}{2},1\\2,\frac{5}{3}\end{matrix};16x^2\right]$$

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$${}_2F_1\left[\begin{matrix}\frac{1}{2},1\\2\end{matrix};4x\right] = \frac{1-\sqrt{1-4x}}{2x}$$

$${}_3F_2\left[\begin{matrix}1/2,\sqrt{2}+1,-\sqrt{2}+1\\\sqrt{2},-\sqrt{2}\end{matrix};4x\right] = \frac{(7x-1)(2x-1)}{(1-4x)^{5/2}}$$

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Schwarz' Classification (Geometry & Algebra)

- **[Schwarz 1873]**: Classification of all algebraic Gaussian hypergeometric functions:

$$F(x) = {}_2F_1([a_1, a_2], [b_1]; x),$$

with rational parameters $a_1, a_2, b_1 \in \mathbb{Q} \setminus \mathbb{Z}$.

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- Write $(\lambda, \mu, \nu) = (1 - b_1, b_1 - a_1 - a_2, a_2 - a_1)$.

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- $F(x)$ is algebraic if and only if (λ, μ, ν) appears in Schwarz' list, up to permutations, sign changes and addition of triples of integers with even sum.

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Example

$F(x) = {}_2F_1([-1/2, -1/6], [2/3]; x)$ is algebraic, as $(\lambda, \mu, \nu) = (1/3, 4/3, 1/3)$ and $(-(\nu - 1), \lambda, \mu - 1) = (2/3, 1/3, 1/3)$ is in the list.

No.	λ''	μ''	ν''	Inhalt / π	Polyeder
I.	$\frac{1}{2}$	$\frac{1}{2}$	ν	ν	Regelmässige Doppelpyramide
II.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6} = A$	Tetraeder
III.	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3} = 2A$	
IV.	[illegible]	[illegible]	[illegible]	[illegible]	

Landau-Errera Criterion (Arithmetic)

[Landau 1904, 1911]: Necessary condition for algebraicity of Gaussian hypergeometric functions (based on Eisenstein's criterion):

Theorem (Landau)

Let $F(x) = {}_2F_1([a_1, a_2], [b_1]; x)$ *with* $a_1, a_2, b_1, a_1 - b_1, a_2 - b_1 \notin \mathbb{Z}$ *and* N *their common denominator. Then* $F(x)$ *is almost integral if and only if for all* $1 \leq \lambda \leq N$ *coprime to* N *we have*

$$\langle \lambda a_1 \rangle < \langle \lambda b_1 \rangle < \langle \lambda a_2 \rangle \quad \text{or} \quad \langle \lambda a_2 \rangle < \langle \lambda b_1 \rangle < \langle \lambda a_1 \rangle, \tag{$\star$}$$

where $\langle \cdot \rangle$ *denotes the fractional part.*

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where $\langle \cdot \rangle$ denotes the fractional part.

Theorem (Errera, 1913)

Condition ($\star$) for all $1 \leq \lambda \leq N$ coprime to N is equivalent to Schwarz' classification.

Christol's Interlacing Criterion

[Christol 1986]: Study of diagonals of multivariate rational functions
(Christol's conjecture).

A by-product classification *à-la-Landau* of almost integral hypergeometric functions.

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A by-product classification *à-la-Landau* of almost integral hypergeometric functions.

Idea: Counting multiplicities of prime numbers $\mathfrak{p}$ in

$$\frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_{p-1})_n \; n!}$$

amounts to the $\mathfrak{p}$-adic evaluation of elements of the arithmetic progressions $r_i + k \cdot N$ and $s_i + k \cdot N$, where $a_i = r_i/N$ and $b_i = s_i/N$.

Christol's Interlacing Criterion

Let $\langle\cdot\rangle : \mathbb{R} \to (0,1]$ as the fractional part, where integers are assigned 1.
Define $\preceq$ on $\mathbb{R}^2$ via $a \preceq b$ if $\langle a\rangle < \langle b\rangle$ or $\langle a\rangle = \langle b\rangle$ and $a \geq b$.

Theorem (Christol, 1986)

Let $a_j, b_k \notin -\mathbb{N}$ be rational numbers and

$$F(x) := {}_pF_{p-1}\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{p-1} \end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_{p-1})_n} \cdot \frac{x^n}{n!}.$$

Denote by N the least common denominator of a_j, b_k, and set $b_p = 1$. Then $F(x)$ is almost integral if and only if for all $1 \leq \lambda \leq N$ with $\gcd(\lambda, N) = 1$ we have for all $1 \leq k \leq p$ that

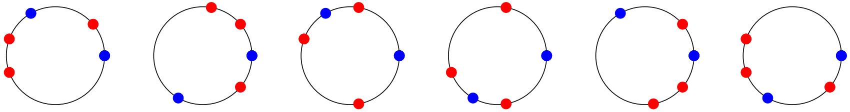
$$|\{\lambda a_j \preceq \lambda b_k : 1 \leq j \leq p\}| - |\{\lambda b_j \preceq \lambda b_k : 1 \leq j \leq p\}| \geq 0.$$

Christol's Interlacing Criterion graphically

Draw the sets $\{\exp(2\pi i\lambda a_j)\}$ in red and $\{\exp(2\pi i\lambda b_k)\}$ in blue on the unit circle for all $1 \le \lambda \le N$ with $\gcd(\lambda, N) = 1$. Then $F(x)$ is almost integral iff there are always at least as many red as blue points going counter-clockwise starting after 1.

Example

${}_3F_2([1/9, 4/9, 5/9], [1/3, 1]; x) = 1 + \frac{20}{3^5}x + \frac{2275}{3^{10}}x^2 + \frac{3124550}{3^{17}}x^3 + \cdots$ is almost integral.



Beukers-Heckman Interlacing Criterion

Theorem (Christol 1986, Beukers-Heckman 1989, Katz 1990)

Let $a_j, b_k \notin -\mathbb{N}$ be rational numbers such that $a_j - b_k, a_j \notin \mathbb{Z}$ and

$$F(x) := {}_pF_{p-1}\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{p-1} \end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_{p-1})_n} \cdot \frac{x^n}{n!}.$$

Denote by N the least common denominator of a_j, b_k, and set $b_p = 1$. Then $F(x)$ is algebraic iff for all $1 \leq \lambda \leq N$ with $\gcd(\lambda, N) = 1$ we have for all $1 \leq k \leq p$ that

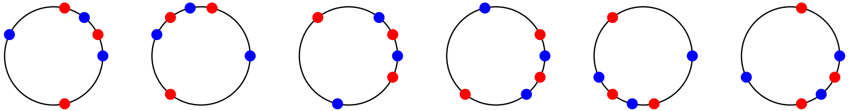
$$|\{\lambda a_j \preceq \lambda b_k : 1 \leq j \leq p\}| - |\{\lambda b_j \preceq \lambda b_k : 1 \leq j \leq p\}| = 0. \qquad \text{(IC)}$$

In other words, $F(x)$ is algebraic, if and only if the sets $\{2\pi i \lambda a_j\}$ and $\{2\pi i \lambda b_k\}$ interlace on the unit circle for all λ.

Beukers-Heckman Interlacing Criterion

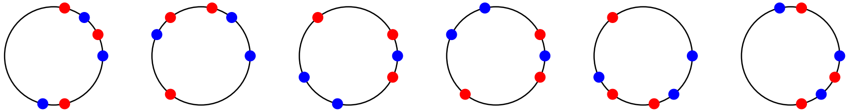
Example

$F(x) = {}_3F_2([1/14, 3/14, 11/14], [1/7, 3/7]; x)$ is algebraic:



Example

$F(x) = {}_3F_2([1/14, 3/14, 11/14], [1/7, 5/7]; x)$ is transcendental:



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- [Beukers-Heckman 1989] has the additional assumption $a_j - b_k, a_j \notin \mathbb{Z}$.
- Assuming $a_j - b_k, a_j \notin \mathbb{Z}$, Christol's criterion is satisfied for $F(x)$ if and only if Beukers-Heckman interlacing holds.

PROPOSITION 3 : *Toute fonction hypergéométrique F réduite et de hauteur 1 est globalement bornée si et seulement si, pour tout Δ tel que $(\Delta, N) = 1$, les nombres $\exp(2i\pi\Delta a_i)$ et $\exp(2i\pi\Delta b_i)$ sont entrelacés sur le cercle unité.*

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- Assuming $a_j - b_k, a_j \notin \mathbb{Z}$, Christol's criterion is satisfied for $F(x)$ if and only if it is satisfied for **all** solutions of the hypergeometric differential equation.
- Under the p-curvature conjecture and $a_j - b_k, a_j \notin \mathbb{Z}$, the criteria are equivalent.

COROLLAIRE (modulo la conjecture de GROTHENDIECK si s>2) : *Une fonction hypergéométrique de hauteur 1 est globalement bornée si et seulement si elle est algébrique.*

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- Hypergeometric equations are a special case of a class of equations for which the conjecture was solved by [Katz 1972].

Beukers-Heckman 1989

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- Analytic continuation of the solutions of the hypergeometric ODE around its singularities $0, 1, \infty$ yields the monodromy group.
- It is finite if and only if all solutions are algebraic.
- There is a Hermitian form invariant under the monodromy group, which is positive definite if and only if the parameters interlace on the unit circle.

Beukers-Heckman 1989

- Analytic continuation of the solutions of the hypergeometric ODE around its singularities $0, 1, \infty$ yields the monodromy group.
- It is finite if and only if all solutions are algebraic.
- There is a Hermitian form invariant under the monodromy group, which is positive definite if and only if the parameters interlace on the unit circle.
- The monodromy group is discrete and contained in the unitary group, which is compact, if and only if interlacing holds.
- Complete classification of all possible resulting finite monodromy groups.

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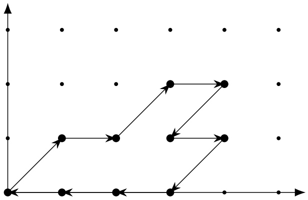
Table 8.3. (continued)

No.	Dimension	Parameter set		Field of definition	Group
2	3	$\frac{3}{14}\ \frac{5}{14}\ \frac{13}{14}$;	$0\ \frac{1}{3}\ \frac{2}{3}$		
3			$0\ \frac{1}{4}\ \frac{3}{4}$	$\mathbb{Q}(\sqrt{-7})$	ST 24
4			$\frac{1}{7}\ \frac{2}{7}\ \frac{4}{7}$		
5		$0\ \frac{1}{5}\ \frac{4}{5}$;	$\frac{1}{6}\ \frac{1}{2}\ \frac{5}{6}$	$\mathbb{Q}(\sqrt{5})$	ST 23
6			$\frac{1}{10}\ \frac{1}{2}\ \frac{9}{10}$		
7		$\frac{1}{6}\ \frac{11}{30}\ \frac{29}{30}$;	$0\ \frac{1}{5}\ \frac{4}{5}$	$\mathbb{Q}(\omega, \sqrt{5})$	ST 27
8			$0\ \frac{1}{4}\ \frac{3}{4}$		

Example: Gessel Excursions

Gessel walks: Lattice walks in the quarter plane with step set $\{\rightarrow, \leftarrow, \nearrow, \swarrow\}$.



Consider the generating function

$$G(x) = \sum_{n \geq 0} g_n x^n$$

of excursions of length n.

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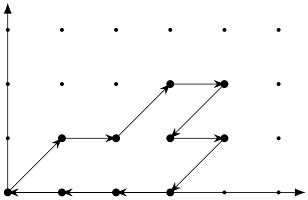

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Theorem (Kauers-Koutschan-Zeilberger, Bousquet-Mélou, Bostan-Kurkova-Raschel)

$$G(x) = \sum_{n \geq 0} \frac{(5/6)_n (1/2)_n}{(2)_n (5/3)_n} 16^n x^{2n} = {}_3F_2\left[\begin{matrix} \frac{5}{6}, \frac{1}{2}, 1 \\ 2, \frac{5}{3} \end{matrix}; 16x^2\right].$$

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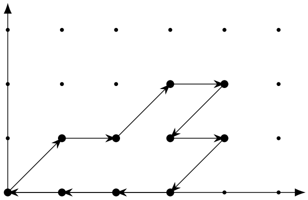

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Question: Is $G(x)$ algebraic?

Is $G(x) = {}_3F_2([5/6, 1/2, 1], [2, 5/3]; 16x^2)$ algebraic?

- Direct application of the interlacing criterion is not possible, as $a_3 = 1 \in \mathbb{Z}$.

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- Trick: Use identities for hypergeometric functions:

$$G(x) = \frac{1}{2x^2}\left({}_2F_1\left[\begin{matrix}-1/2, -1/6\\ 2/3\end{matrix}; 16x^2\right] - 1\right),$$

is algebraic by [Schwarz 1873] or [Landau-Errera] or [Beukers-Heckman 1989].

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- Algebraicity of $G(x)$ was overlooked until [Bostan-Kauers 2010]; [Bostan 2017].
- The minimal polynomial $P(x, \boldsymbol{y})$ of $\boldsymbol{G(x)}$ is

$$\begin{aligned}P(x, \boldsymbol{y}) = 27x^{14}\boldsymbol{y}^8 + 108x^{12}\boldsymbol{y}^7 + 189x^{10}\boldsymbol{y}^6 + 189x^8\boldsymbol{y}^5 - 9x^6(32x^4 + 28x^2 - 13)\boldsymbol{y}^4\\ -9x^4(64x^4 + 56x^2 - 5)\boldsymbol{y}^3 - 2x^2(256x^6 - 312x^4 + 156x^2 - 5)\boldsymbol{y}^2\\ -(32x^2 - 1)(4x^2 - 6x + 1)(4x^2 + 6x + 1)\boldsymbol{y} - 256x^6 - 576x^4 + 48x^2 - 1\end{aligned}$$

Example 2: Irrational Parameters

Consider the innocent recursion

$$u_{n+1} = \frac{2(2n+1)(n^2+2n-1)}{(n+1)(n^2-2)} u_n, \quad u_0 = 1.$$

The generating function

$$\sum_{n\geq 0} u_n x^n = {}_3F_2\left[\begin{matrix} 1/2, \sqrt{2}+1, -\sqrt{2}+1 \\ \sqrt{2}, -\sqrt{2} \end{matrix}; 4x\right] = \frac{(7x-1)(2x-1)}{(1-4x)^{5/2}}$$

is algebraic, but the criteria are not applicable.

Goal

The interlacing criteria treat the case of $a_j, b_k \in \mathbb{Q} \setminus \{-\mathbb{N}\}$ with $a_j - b_k, a_j \notin \mathbb{Z}$.

Aim: An easy-to-use algebraicity criterion to account for irrational parameters and integer differences.

Examples

$$ {}_2F_1\left[\begin{matrix}1,1\\2\end{matrix};x\right] = -\frac{\log(1-x)}{x} $$

$$ {}_2F_1\left[\begin{matrix}\frac{1}{2},1\\2\end{matrix};4x\right] = \frac{1-\sqrt{1-4x}}{2x} $$

$$ {}_3F_2\left[\begin{matrix}1/2,\sqrt{2}+1,-\sqrt{2}+1\\\sqrt{2},-\sqrt{2}\end{matrix};4x\right] = \frac{(7x-1)(2x-1)}{(1-4x)^{5/2}} $$

$$ {}_6F_5\left[\begin{matrix}\frac{1}{4},\frac{1}{2},\frac{3}{4},3,3,1\\\frac{1}{3},\frac{2}{3},4,2,2\end{matrix};\frac{256}{27}x\right] $$

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Some Definitions

- Define
$$\mathcal{F}\left[\begin{matrix} c_1,\dots,c_r \\ d_1,\dots,d_s \end{matrix};x\right] := \sum_{n\geq 0} \frac{(c_1)_n\cdots(c_r)_n}{(d_1)_n\cdots(d_s)_n}x^n.$$

- Note that
$${}_pF_q\left[\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix};x\right] = \mathcal{F}\left[\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q,1 \end{matrix};x\right],$$
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- $F(x)$ is contracted if $c_j - d_k \notin \mathbb{N}$. $F(x)$ is reduced if $c_j - d_k \notin \mathbb{Z}$.

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- $F(x)$ is contracted if $c_j - d_k \notin \mathbb{N}$. $F(x)$ is reduced if $c_j - d_k \notin \mathbb{Z}$.
- The contraction $F^c(x)$ of $F(x)$ is obtained by removing pairs of parameters (c_j, d_k) with minimal difference $c_j - d_k \in \mathbb{N}$.
  If $F(x)$ is given as ${}_pF_q$, convert to $\mathcal{F}$ first.

Examples

$$ {}_4F_3\left[\begin{matrix}\frac{1}{3},\frac{1}{2},2,4\\ \frac{3}{2},3,1\end{matrix};x\right]^c = \mathcal{F}\left[\begin{matrix}\frac{1}{3},\frac{1}{2},2,4\\ \frac{3}{2},3,1,1\end{matrix};x\right]^c = \mathcal{F}\left[\begin{matrix}\frac{1}{3},\frac{1}{2}\\ \frac{3}{2},1\end{matrix};x\right] $$

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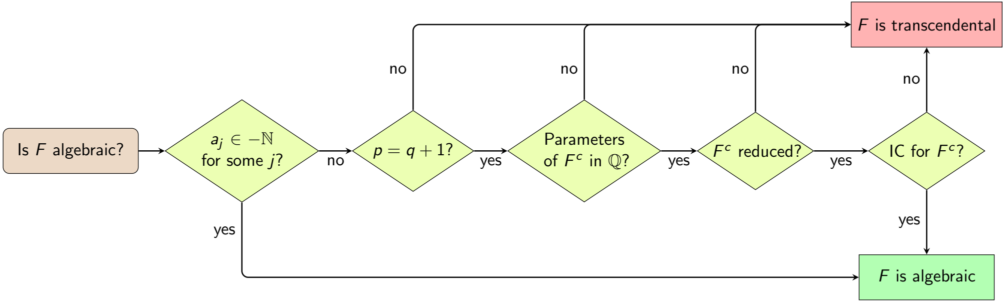
The Criterion

Theorem (Fürnsinn-Y. 2024)

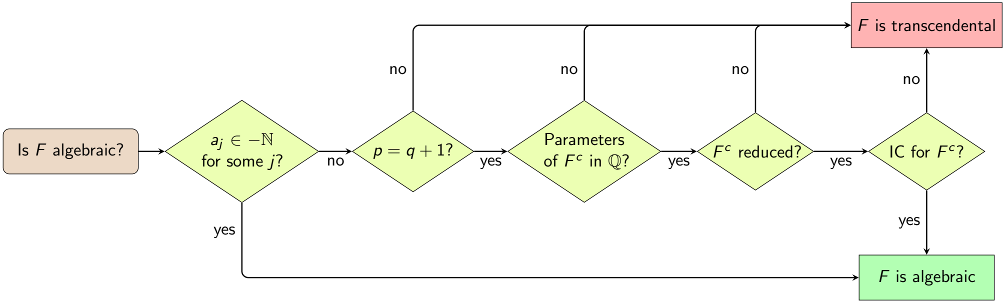
For any hypergeometric function $F(x) = {}_pF_q([a_1, \ldots, a_p], [b_1, \ldots, b_q]; x) \in \mathbb{Q}[\![x]\!]$ the following decision tree answers the question whether it is algebraic.



[Caruso, Fürnsinn, 2026]: Implementation (many other algorithms).

Main steps of the proof

- Contraction preserves algebraicity.
- Algebraic contracted hypergeometric functions have rational parameters.
- Algebraic contracted hypergeometric functions are reduced.



Contraction Preserves Algebraicity

Proposition

The hypergeometric function $F(x)$ is algebraic if and only if $F^c(x)$ is algebraic.

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Proof sketch.

Contiguous relation:
$$(\theta + a_1)F(x) = a_1 \cdot {}_pF_{p-1}\left[\begin{matrix} a_1+1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_{p-1}\end{matrix}; x\right].$$

Euclidean division of the hypergeometric operator $H(\theta)$ by $(\theta + \alpha)$:
$$H(\theta) = Q(\theta)(\theta+\alpha) + H(-\alpha), \quad H(-\alpha) = x\prod_{j=1}^{p}(-\alpha + a_j) + \alpha \prod_{k=1}^{p-1}(-\alpha + b_k - 1).$$

Thus, $0 = Q(\theta)(\theta+\alpha)F(x) + H(-\alpha)F(x)$. $H(-\alpha)$ vanishes iff $\alpha \in \{a_1, \ldots, a_p\} \cap \{b_1 - 1, \ldots, b_{p-1} - 1, 0\}$. Otherwise, $(\theta+\alpha)F(x)$ is algebraic iff $F(x)$ is algebraic. Iterating $\Rightarrow$ F algebraic iff F^c algebraic. □

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Alternative proof: [André 1989]: $F(x)$ algebraic iff $F'(x)$ algebraic & almost integral.

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Lemma (Folklore, Singer 1980)

If $f(x)$ is algebraic then all solutions of $L_f^{\min}$ are algebraic.

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Lemma (Folklore, Singer 1980)

If $f(x)$ is algebraic then all solutions of $L_f^{\min}$ are algebraic.

Proposition

If $F(x)$ is contracted and has irrational parameters, it is transcendental.

Alternative proof: [Galočkin 1981] classification of hypergeometric G-functions.

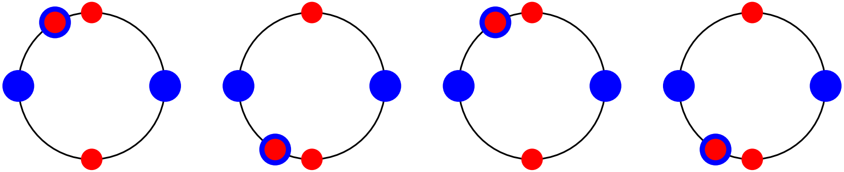
Algebraic Contracted Hypergeometric Functions Are Reduced

Proposition

If $F(x)$ is contracted but not reduced, then $F(x)$ is transcendental.

We prove that $F(x)$ is not almost integral using [Christol's criterion].
Proof "by example": Let

$$F(x) = {}_3F_2\left[\begin{matrix}\frac{1}{4}, \frac{3}{4}, \frac{1}{3} \\ \frac{1}{2}, \frac{4}{3}\end{matrix}; x\right] \quad \Rightarrow N = 12, \lambda \in \{1, 5, 7, 11\}$$



Recall: $\langle \cdot \rangle$ is the fractional part and $a \preceq b$ if $\langle a \rangle < \langle b \rangle$ or $\langle a \rangle = \langle b \rangle$ and $a \geq b$.

Example 1

Let: $$u_{n+1} = \frac{(14n+1)(14n+3)(14n+11)\left(n^2+2n+4\right)}{56(7n+1)(7n+3)(n+3)\left(n^2+3\right)} u_n, \quad u_0 = 1$$

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Generating function:

$$f(x) = \mathcal{F}\left[\begin{matrix}\frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1+i\sqrt{3}, 1-i\sqrt{3} \\ \frac{1}{7}, \frac{3}{7}, i\sqrt{3}, -i\sqrt{3}, 3\end{matrix}; x\right] = {}_6F_5\left[\begin{matrix}\frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1+i\sqrt{3}, 1-i\sqrt{3}, 1 \\ \frac{1}{7}, \frac{3}{7}, i\sqrt{3}, -i\sqrt{3}, 3\end{matrix}; x\right].$$

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Contraction has rational parameters and is reduced:

$$f^c(x) = \mathcal{F}\left[\begin{matrix}\frac{1}{14}, \frac{3}{14}, \frac{11}{14} \\ \frac{1}{7}, \frac{3}{7}, 3\end{matrix}; x\right] = {}_4F_3\left[\begin{matrix}\frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1 \\ \frac{1}{7}, \frac{3}{7}, 3\end{matrix}; x\right].$$

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$$\text{Let:}\quad u_{n+1} = \frac{(14n+1)(14n+3)(14n+11)\left(n^2+2n+4\right)}{56(7n+1)(7n+3)(n+3)\left(n^2+3\right)}u_n, \quad u_0 = 1$$

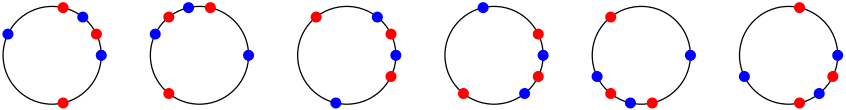
Generating function:

$$f(x) = \mathcal{F}\left[\begin{matrix}\frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1+i\sqrt{3}, 1-i\sqrt{3} \\ \frac{1}{7}, \frac{3}{7}, i\sqrt{3}, -i\sqrt{3}, 3\end{matrix}; x\right] = {}_6F_5\left[\begin{matrix}\frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1+i\sqrt{3}, 1-i\sqrt{3}, 1 \\ \frac{1}{7}, \frac{3}{7}, i\sqrt{3}, -i\sqrt{3}, 3\end{matrix}; x\right].$$

Contraction has rational parameters and is reduced:

$$f^c(x) = \mathcal{F}\left[\begin{matrix}\frac{1}{14}, \frac{3}{14}, \frac{11}{14} \\ \frac{1}{7}, \frac{3}{7}, 3\end{matrix}; x\right] = {}_4F_3\left[\begin{matrix}\frac{1}{14}, \frac{3}{14}, \frac{11}{14}, 1 \\ \frac{1}{7}, \frac{3}{7}, 3\end{matrix}; x\right].$$

$f^c(x)$ is algebraic by the interlacing criterion, thus so is $f(x)$.



Example 2

$$u_n = \frac{3}{2}\binom{4n}{n}\frac{n+2}{(n+1)(n+3)}.$$

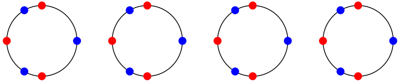
Generating function:

$$f(x) = {}_6F_5\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 3, 3, 1 \\ \frac{1}{3}, \frac{2}{3}, 4, 2, 2\end{matrix}; \frac{256}{27}x\right]$$

Contraction:

$$f^c(x) = {}_4F_3\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \\ \frac{1}{3}, \frac{2}{3}, 4\end{matrix}; \frac{256}{27}x\right]$$

Interlacing criterion: $f(x)$ algebraic.



Example 2

$$u_n = \frac{3}{2}\binom{4n}{n}\frac{n+2}{(n+1)(n+3)}.$$

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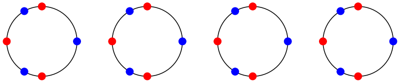
Contraction:

$$f^c(x) = {}_4F_3\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\\ \frac{1}{3}, \frac{2}{3}, 4\end{matrix}; \frac{256}{27}x\right]$$

Interlacing criterion: $f(x)$ algebraic.



$$v_n = \frac{3}{2}\binom{4n}{n}\frac{n+2}{(n+1)^2}.$$

Generating function:

$$g(x) = {}_6F_5\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 3, 1, 1\\ \frac{1}{3}, \frac{2}{3}, 2, 2, 2\end{matrix}; \frac{256}{27}x\right]$$

Contraction:

$$g^c(x) = {}_5F_4\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1\\ \frac{1}{3}, \frac{2}{3}, 2, 2\end{matrix}; \frac{256}{27}x\right]$$

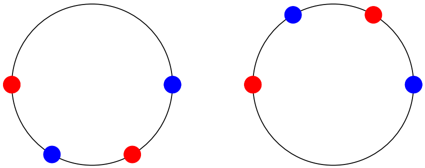
Not reduced: $g(x)$ not algebraic.

Example 3 – Gessel Revisited

Recall the generating function of Gessel excursions

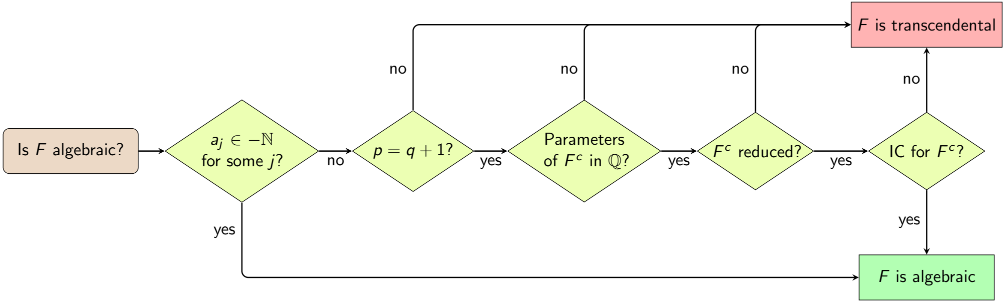
$$G(x) = {}_3F_2\left[\begin{matrix}\frac{5}{6}, \frac{1}{2}, 1\\ 2, \frac{5}{3}\end{matrix}; 16x^2\right] = \mathcal{F}\left[\begin{matrix}\frac{5}{6}, \frac{1}{2}\\ 2, \frac{5}{3}\end{matrix}; x\right].$$

$G(x)$ is contracted, reduced, has rational parameters, satisfies the interlacing criterion:



The End

Thank you for your attention!



Bonus: Algebraicity of $F(x)$ vs $F'(x)$

Corollary

For a hypergeometric function $F(x) \in \mathbb{Q}[\![x]\!]$,

$$F(x) \text{ is algebraic} \iff F'(x) \text{ is algebraic.}$$

Bonus: Algebraicity of $F(x)$ vs $F'(x)$

Corollary

For a hypergeometric function $F(x) \in \mathbb{Q}[[x]]$,

$$F(x) \text{ is algebraic } \iff F'(x) \text{ is algebraic.}$$

$\log(1-x)$ is not hypergeometric and the derivative of $\log(1-x)/x$ is not algebraic.

Proof.

"$\Rightarrow$": Derivatives of algebraic series are algebraic.
"$\Leftarrow$": Use the identity

$$\frac{\mathrm{d}}{\mathrm{d}x} {}_pF_q\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; x\right] = \frac{a_1 \cdots a_p}{b_1 \cdots b_q} \cdot {}_pF_q\left[\begin{matrix} a_1+1, \dots, a_p+1 \\ b_1+1, \dots, b_q+1 \end{matrix}; x\right].$$

□

Bonus: Galočkin's Theorem on Irrational Parameters

Theorem (Galočkin 1981)

A hypergeometric function ${}_pF_q([a_1, \ldots, a_p], [b_1, \ldots, b_q]; x)$ is a G-function if and only if its irrational parameters can be arranged in pairs (a_{j_ν}, b_{k_ν}) with $a_{j_\nu} - b_{k_\nu} \in \mathbb{N}$.

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In other words: A hypergeometric function is a G-function iff its contraction has only rational parameters.