The generating function of Yang-Zagier numbers is algebraic[1]

SIAM AG21

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[1] Joint work with Alin Bostan and Jacques-Arthur Weil.

Two sequences

$$(a_n)_{n\geq 0} = (1\,,\ -48300\,,\ 7981725900\,,\ -1469166887370000, \ldots)$$
$$(b_n)_{n\geq 0} = (1\,,\ -144900\,,\ 88464128725\,,\ -62270073456990000\,,\ \ldots)$$

Origin of a_n and b_n

- In Arithmetic and Topology of Differential Equations, 2018 by Don Zagier:

$$c_{n-3} + 20\left(4500n^2 - 18900n + 19739\right)c_{n-2} + 80352000n(5n-1)(5n-2)(5n-4)c_n + 25\left(2592000n^4 - 16588800n^3 + 39118320n^2 - 39189168n + 14092603\right)c_{n-1} = 0,$$

with initial terms $c_0 = 1, c_1 = -161/(2^{10} \cdot 3^5)$ and $c_2 = 26605753/(2^{23} \cdot 3^{12} \cdot 5^2)$.

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- Recursion comes from an integral over a moduli space ("topological ODE") [Bertola, et al, 2015].

Problem (Zagier, 2018)

Find $(u, v) \in \mathbb{Q}^* \times \mathbb{Q}^*$ such that $c_n \cdot (u)_n \cdot (v)_n \cdot w^n \in \mathbb{Z}$ for some $w \in \mathbb{Z}^*$.

$$(u)_n := u\cdot(u+1)\cdots(u+n-1).$$

- [Yang and Zagier]: $a_n = c_n \cdot (3/5)_n \cdot (4/5)_n \cdot (2^{10}\cdot 3^5\cdot 5^4)^n \in \mathbb{Z}$,
- [Dubrovin and Yang]: $b_n = c_n \cdot (2/5)_n \cdot (9/10)_n \cdot (2^{12}\cdot 3^5\cdot 5^4)^n \in \mathbb{Z}$.

Mystery of a_n and b_n

- "Yang and I found a formula showing that the numbers a_n are integers of exponential growth and hence can be expected to have a generating series that is a **period**, although we have not succeeded in finding it" – [Zagier, 2018]
- "Dubrovin and Yang found that the numbers b_n are *also* integral and that in this case the generating function is not only of Picard-Fuchs type, but is actually **algebraic**!" – [Zagier, 2018]
- "So this is a very mysterious example [...] of numbers defined by recursions with polynomial coefficients." – [Zagier, 2018]
- "My presumed arithmetic intuition [...] was entirely broken" – [Wadim Zudilin]

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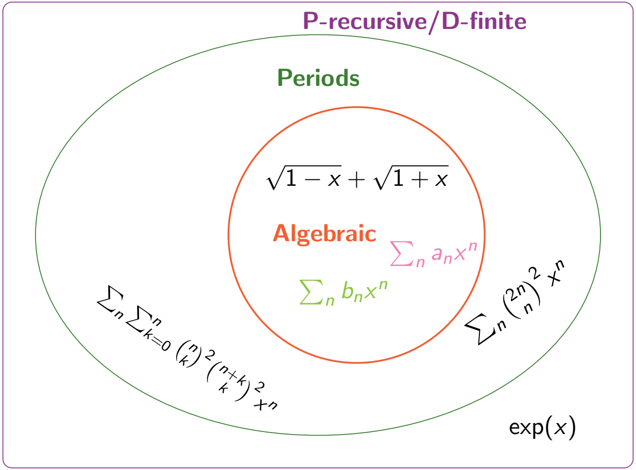
Problem

Investigate the nature of $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and similar sequences.

Theorem (Bostan,Weil,Y.)

The generating functions of both $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are algebraic.

Definitions and interactions



A sequence $(u_n)_{n\geq 0}$ is **P-recursive**, if it satisfies a linear recurrence with polynomial coefficients:

$$c_d(n)u_{n+d} + \cdots + c_0(n)u_n = 0.$$

$u_n = 1/n!$ satisfies $nu_n = u_{n-1}$.

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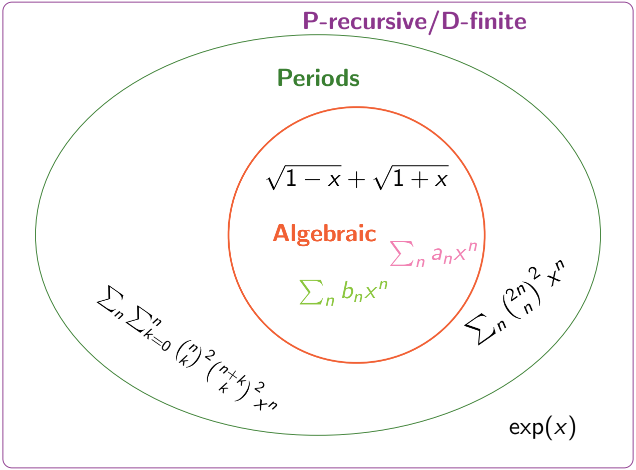

A power series $f(x) \in \mathbb{Q}[\![x]\!]$ is **D-finite** if it satisfies a linear differential equation with polynomial coefficients:

$$p_n(x) f^{(n)}(x) + \cdots + p_0(x) f(x) = 0.$$

This equation can be rewritten: $L \cdot f = 0$,

$$L = p_n(x)\partial^n + \cdots + p_0(x) \in \mathbb{Q}(x)[\partial],$$

where $\partial := \frac{\mathrm{d}}{\mathrm{d}x}$.

$\exp(x)$ satisfies $\exp'(x) = \exp(x)$.
$L = \partial - 1$.

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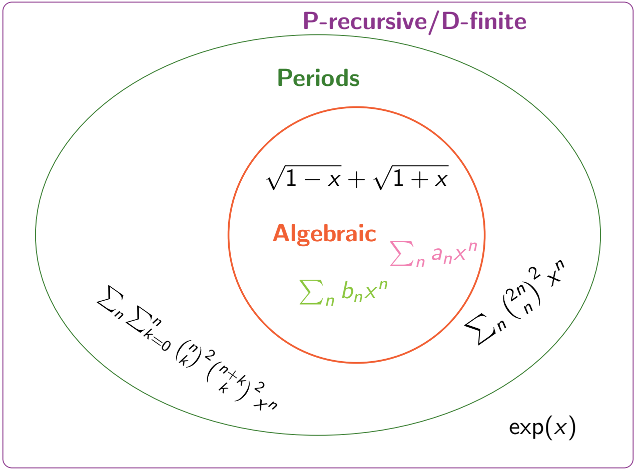

A power series $f(x) \in \mathbb{Q}[\![x]\!]$ is called a **Period function** if it is an integral over a submanifold of a differential form depending algebraically on x.

$$p(e) = 4\int_0^1 \sqrt{\frac{1-e^2t^2}{1-t^2}}\mathrm{d}t$$

$$= 4\oiint \frac{\mathrm{d}u\mathrm{d}v}{1-\frac{1-e^2u^2}{(1-u^2)v^2}} \text{ and}$$

$$((e-e^3)\partial^2 + (1-e^2)\partial + e)\cdot p = 0,$$

$$p(e) = 2\pi - \frac{\pi}{2}e^2 - \frac{3\pi}{32}e^4 - \cdots .$$

Back to a_n and b_n

- $(a_n)_n$ and $(b_n)_n$ are **P-recursive** sequences $\Rightarrow$ generating functions are **D-finite**.

$$L_a = 1800x(7x-62)\left(x^2+50x+20\right)\partial^2 + 720(42x^3+173x^2-14230x-620)\partial + 6048x^2 - 139453x - 249550 \in \mathbb{Q}(x)[\partial],$$

$$\begin{aligned} L_b = {} & 90000x^3(2911x+310)\left(x^2+50x+20\right)\partial^4 \\ & + 18000x^2\left(154283x^3+5185005x^2+1675710x+142600\right)\partial^3 \\ & + 50x\left(147290778x^3+2740219655x^2+566777510x+37497600\right)\partial^2 \\ & + 5\left(919899288x^3+5629046605x^2+1348939210x+10713600\right)\partial \\ & + 18\left(13937868x^2-1076845x+1247750\right) \in \mathbb{Q}(x)[\partial]. \end{aligned}$$

- The generating functions of $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ solve $L_a \cdot y = 0$ and $L_b \cdot y = 0$

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Stanley's problem (1980)

Given a **D-finite** series how to prove or disprove that it is **algebraic**?

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- New practical algorithm for **disproving** **algebraicity** [Bostan, Rivoal, Salvy, 2021].

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- New practical algorithm for **disproving** **algebraicity** [Bostan, Rivoal, Salvy, 2021].
- Several tests for justifying **algebraicity** based on **conjectures** or **numerics**: work well in practice but do not provide proofs.
- Differential Galois theory based method sometimes efficient proving **algebraicity**.

Grothendieck-Katz conjecture: "testing" **algebraicity**

- $L \cdot y = 0$ is equivalent to $Y' = A(x)Y$, where $A(x) \in M^{n\times n}(k)$ and $k = \mathbb{Q}(x)$.
- The p-curvature is the matrix $A_p(x) \in \mathbb{Q}(x)$, where

$$A_0(x) = \mathrm{Id}_n, \quad \text{and} \quad A_{\ell+1}(x) = A'_\ell(x) + A_\ell(x)A(x) \quad \text{for} \quad \ell \geq 0.$$

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Conjecture (Grothendieck 1960's; Katz, 1972)

All solutions of $Y' = A(x)Y$ are algebraic if and only if $A_p = 0 \bmod p$ for almost all primes p.

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Conjecture (Grothendieck 1960's; Katz, 1972)

All solutions of $Y' = A(x)Y$ are algebraic if and only if $A_p = 0$ mod p for almost all primes p.

- A_p mod p can be efficiently computed [Bostan, Caruso, Schost, 2015].
- L_a and L_b have 0 p-curvature for all primes $11 \leq p \leq 97$.

Monodromy group: quantifying **algebraicity**

- $L \cdot y = 0$ for $L \in \mathbb{Q}(x)[\partial]$ has $n = \mathrm{ord}(L)$ linearly independent solutions.
- Assume $f_1, \ldots, f_n$ are linearly independent solutions at 0. If we analytically continue them along a closed loop in $\mathbb{C}$, we find $\widetilde{f_1}, \ldots, \widetilde{f_n}$ possibly different.
- There exists $M_{\underline{f}} \in \mathrm{GL}(n, \mathbb{C})$ such that

$$\begin{pmatrix} \widetilde{f_1} \\ \vdots \\ \widetilde{f_n} \end{pmatrix} = M_{\underline{f}} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

- The matrices $M_{\underline{f}}$ define the so-called monodromy group M.

Theorem (Singer, Ulmer, 1993)

Let f be a solution of $L \cdot y = 0$. The algebraicity degree of f is equal to the cardinality of the orbit of f under the action of M.

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- M can be efficiently computed numerically [Chudnovsky^2, 1987], [van der Hoeven, 1999, 2001], [Mezzarobba, 2010].
- Numerical computations suggest: solutions of L_a and L_b have alg. degree 120.

Differential Galois theory: proving **algebraicity**

- $L \cdot y = 0$ is equivalent to $Y' = A(x)Y$, where $A(x) \in M^{n \times n}(k)$ and $k = \mathbb{Q}(x)$.
- Picard-Vessiot extension: $K = k(U)$, where U is a fundamental solution matrix.
- The differential Galois group G is the group of field automorphisms of K which commute with the derivation and leave all elements of k invariant:

$$G := \mathrm{Aut}_{\partial}(K/k) = \{\sigma \in \mathrm{Aut}(K) \colon \sigma|_k \equiv \mathrm{id}_k \text{ and } \sigma \circ \partial \equiv \partial \circ \sigma\}.$$

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- G is a linear algebraic subgroup of $\mathrm{GL}_n(\mathbb{Q})$.
- G stabilizes the ideal of differential relations between solutions. Moreover:

Theorem (Kolchin, 1948)

$L \cdot y = 0$ has a basis of algebraic solutions if and only if G is finite.

- In practice G is difficult to compute [Hrushovski, 2002], [Feng, 2015], [van der Hoeven, 2007], [Amzallag, Minchenko, Pogudin, 2018], [Sun, 2019].

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- G is the differential Galois group.
- Galois-Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$: Lie algebra of G, i.e. the tangent space of G at id.
- $\mathfrak{g}$ measures the transcendence of K over k:

Theorem (Kolchin, 1948)

If K is the Picard-Vessiot extension of $Y' = A(x)Y$ and $\mathfrak{g} = \mathrm{Lie}(G)$, then

$$\dim_{\mathbb{C}}(G) = \dim_{\mathbb{C}}(\mathfrak{g}) = \mathrm{trdeg}(K/k).$$

- Efficient algorithm for computing $\mathfrak{g}$ [Barkatou, Cluzeau, Di Vizio, Weil, 2020].
- Idea: Compute symmetric powers of L and find **rational solutions** of them. These solutions yield information for $\mathfrak{g}$ via solving a **linear** system.

Toy example

- The operator $L = (4x^2 - 4)\partial^2 + 4x\partial - 1$ has basis of algebraic solutions:
$$\sqrt{1+x} + \sqrt{1-x} \text{ and } \sqrt{1+x} - \sqrt{1-x}.$$
- $L \cdot y = 0$ is equivalent to $Y' = A(x)Y$ where $A(x) = \begin{pmatrix} 0 & 1 \\ \frac{1}{4x^2-4} & \frac{-4x}{4x^2-4} \end{pmatrix}$.

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- If $Y = (y_1, y_2)^t$ is a solution to $Y' = A(x)Y$ then $Y = (y_1^2, 2y_1y_2, y_2^2)^t$ is a solution to the symmetric square system $Y' = A^{(2)}(x)Y$, where now

$$A^{(2)}(x) = \frac{1}{4(x^2-1)} \begin{pmatrix} 0 & 4x^2-4 & 0 \\ 2 & -4x & 8x^2-8 \\ 0 & 1 & -8x \end{pmatrix}.$$

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- It has rational solutions! $F_1 = (4x, 4, x/(x^2-1))^t$, $F_2 = (-4, 0, 1/(x^2-1))^t$.
- If $M \in \mathfrak{g}^{(2)}$ then $MF = 0$ and M comes from a symmetric square. I.e. M satisfies

$$\begin{pmatrix} 2m_{1,1} & m_{1,2} & 0 \\ 2m_{2,1} & m_{1,1}+m_{2,2} & 2m_{1,2} \\ 0 & m_{2,1} & 2m_{2,2} \end{pmatrix} \cdot F_\ell = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad m_{i,j} \in \mathbb{Q}(x), \ell = 1, 2.$$

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- The only solution is $m_{i,j} = 0$. Hence $\mathfrak{g}^{(2)} = \mathfrak{g} = 0$. All solutions of L are algebraic.

The generating sequence of $(b_n)_n$ is algebraic (known to Dubrovin & Yang)

- For L_b same method as in the toy example works.
- $L_b \cdot y = 0$ equivalent to $Y' = A(x)Y$ for $A(x) \in M^{4\times 4}(\mathbb{Q}(x))$.
- The fifth symmetric power $Y' = A^{(5)}(x)Y$ has rational solutions.
- $A^{(5)}(x) \in M^{N\times N}(\mathbb{Q}(x))$, where $N = \binom{4+5-1}{4-1} = 56$.
- Finding the rational solutions takes ≈2 min on a regular PC.
- The corresponding system in $m_{i,j}$ has no non-zero solutions in $\mathbb{Q}(x)$ (≈15 sec).
- $\Rightarrow \mathfrak{g}_b = 0$, therefore L_b has only algebraic solutions.

The generating sequence of $(a_n)_n$ is algebraic (new!)

- For the generating function of $(a_n)_{n\geq 0}$ same method as for $(b_n)_{n\geq 0}$ works.
- The 20th symmetric power has rational solutions ($\approx$ 1 sec).
- $A^{(20)} \in M^{N\times N}(\mathbb{Q}(x))$, where $N = \binom{2+20-1}{2-1} = 21$.
- The corresponding system in $m_{i,j}$ has no non-zero solutions in $\mathbb{Q}(x)$ ($\approx$4 sec).
- $\Rightarrow \mathfrak{g}_a = 0$, therefore L_a has only algebraic solutions.

Experimental mathematics: more similar examples

Problem

Find $(u, v) \in \mathbb{Q}^* \times \mathbb{Q}^*$ such that $c_n \cdot (u)_n \cdot (v)_n \cdot w^n \in \mathbb{Z}$ for some $w \in \mathbb{Z}^*$.

$$(u)_n := u \cdot (u+1) \cdots (u+n-1).$$

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#	u	v	ODE order	degree	#	u	v	ODE order	degree
a_n	3/5	4/5	2	120	f_n	19/60	49/60	4	155520
b_n	2/5	9/10	4	120	g_n	19/60	59/60	4	46080
c_n	1/5	4/5	2	120	h_n	29/60	49/60	4	46080
d_n	7/30	9/10	4	155520	i_n	29/60	59/60	4	155520
e_n	9/10	17/30	4	155520					

- "Test": 0 p-curvatures for primes $< 100 \rightarrow$ expect **algebraic** generating functions.
- Quantify: Guesses for degrees based on numerics.
- Proof: Done: a_n, b_n, c_n. In progress: $d_n, e_n, f_n, g_n, h_n, i_n$.

Summary

- Both sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ have algebraic generating functions, hence they are particular **periods**.
- The Grothendieck-Katz conjecture allows efficient "testing" whether a **D-finite** series is **algebraic**.
- Numerical monodromy group calculations allow efficient quantifying **algebraicity** of **D-finite** series.
- Differential Galois theory allows efficient proving that **D-finite** series is **algebraic**.