

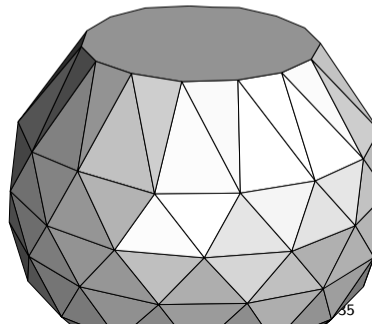
On Rupert's problem¹

PolSys Seminar (Sorbonne University)

Sergey Yurkevich

17th April, 2026

¹Based on joint work with Jakob Steiningger:
arxiv.org/abs/2508.18475 and arxiv.org/abs/2112.13754



Prince Rupert's cube

Fact (Wallis, 1685)

It is possible to cut a hole inside the **unit cube** such that another **unit cube** can pass through this hole.

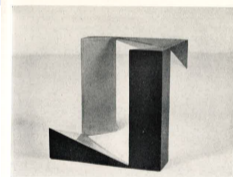
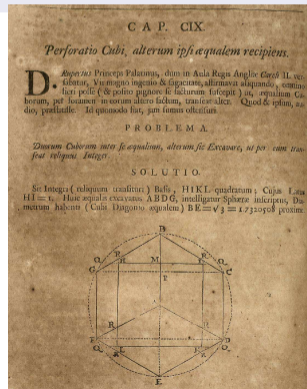
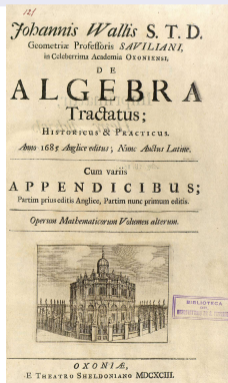
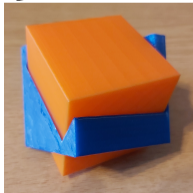
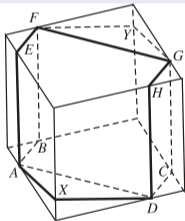


Fig. 4a

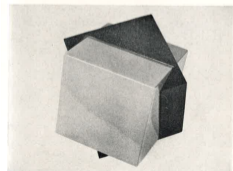
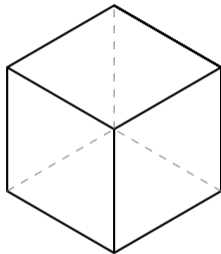
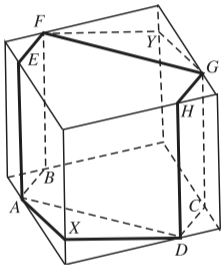
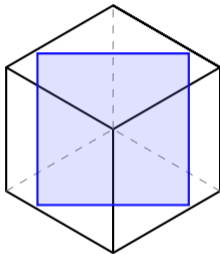
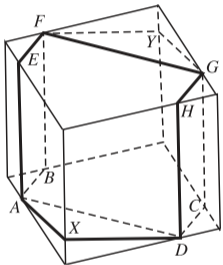


Fig. 4b

Definition of Rupert's problem

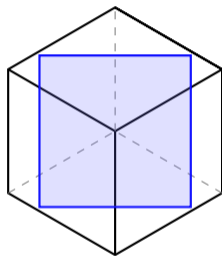
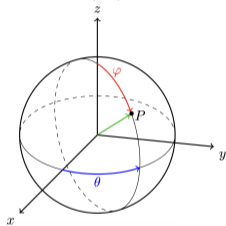


Definition of Rupert's problem



Definition of Rupert's problem

Let $M(\theta, \varphi) : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be an orthogonal projection map in direction $X(\theta, \varphi) \in \mathbb{R}^3$ and $R(\alpha) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the rotation map.



$$X(\theta, \varphi) := (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)^t,$$

$$M(\theta, \varphi) := \begin{pmatrix} -\sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) \cos(\varphi) & -\sin(\theta) \cos(\varphi) & \sin(\varphi) \end{pmatrix},$$

$$R(\alpha) := \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Definition

A point-symmetric polyhedron \mathbf{P} has Rupert's property, if there exist 5 parameters $\alpha, \theta_1, \theta_2 \in [0, 2\pi)$ and $\varphi_1, \varphi_2 \in [0, \pi]$ such that

$$R(\alpha) \circ M(\theta_1, \varphi_1) \mathbf{P} \subset (M(\theta_2, \varphi_2) \mathbf{P})^\circ.$$

Brief history of Rupert's problem

- The Cube is Rupert [conjectured by Prince Rupert, proved by Wallis 1685].
- The “optimal” solution for the Cube [Nieuwland, 1816].

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All convex polyhedra have Rupert's property.

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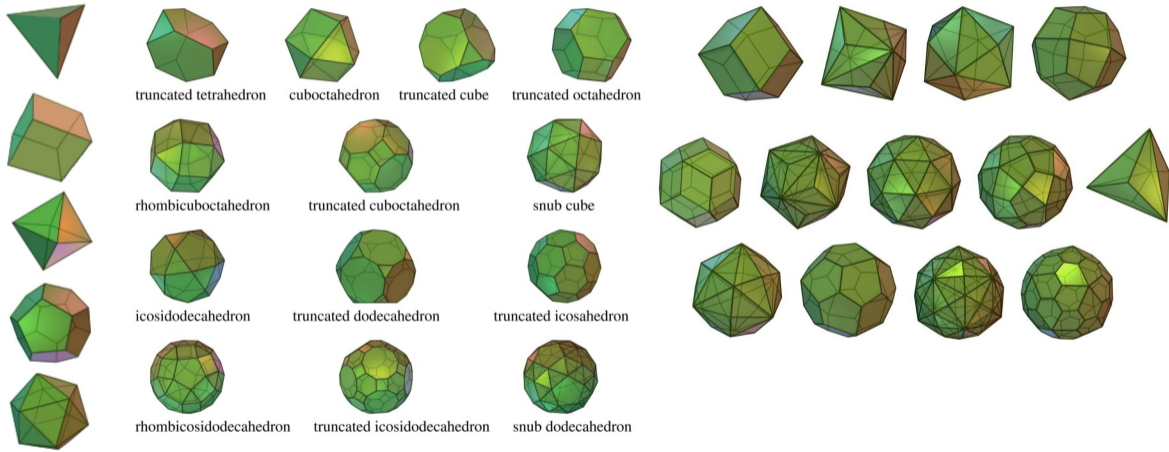
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- 10 of 13 Archimedean solids and many other polyhedra are Rupert. Efficient way for proving Rupert's property. Theoretical algorithm for deciding. [S., Y., 2021].
- 11 of 13 Catalan solids are Rupert, improved optimization [Fredriksson, 2022].
- The **Noperthedron**: a counter example to the conjecture [Steininger, Y., 2025].

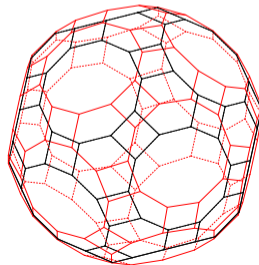
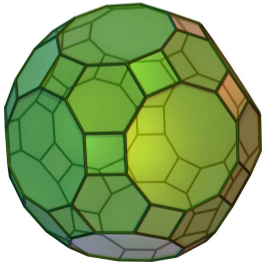
Solids



The Truncated Icosidodecahedron

Theorem (Steininger, Y., 2021)

The Truncated Icosidodecahedron has Rupert's property.



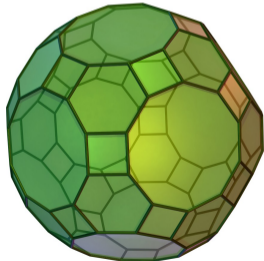
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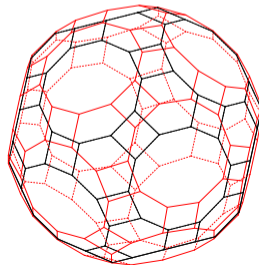
The Truncated Icosidodecahedron has Rupert's property.

Proof.

$\alpha = 0.43584, \theta_1 = 2.77685, \theta_2 = 0.79061, \varphi_1 = 2.09416, \varphi_2 = 2.89674,$
plus some verification of linear inequalities in Maple/SageMath. □



$$R(\alpha) \circ M(\theta_1, \varphi_1)\mathbf{P} \subset (M(\theta_2, \varphi_2)\mathbf{P})^\circ.$$

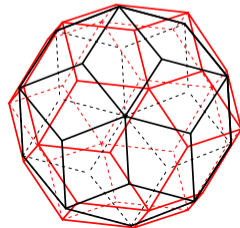
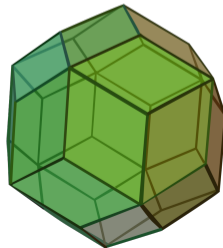
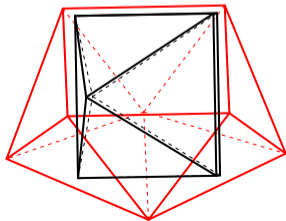
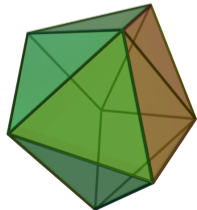


Platonic, Archimedean, Catalan and Johnson Solids

Theorem (Steininger, Y., 2021)

In a few minutes it can be automatically proven that:

- 1** *All 5 Platonic solids are Rupert.*
- 2** *At least 10 out of 13 Archimedean solids are Rupert.*
- 3** *At least 9 out of 13 Catalan solids are Rupert.*
- 4** *At least 82 out of 92 Johnson solids are Rupert.*

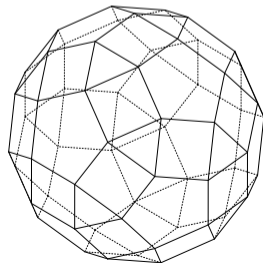


Are all convex polyhedra Rupert?

Conjecture (Jerrard, Wetzel, Yuan, 2017 and Chai, Yuan, Zamfirescu, 2018)

All convex polyhedra have Rupert's property.

- All Platonic solids are Rupert.
- 3 Archimedean solids remain open. One of them is point-symmetric.

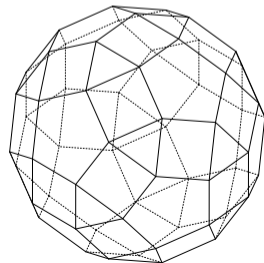


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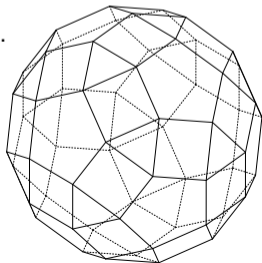


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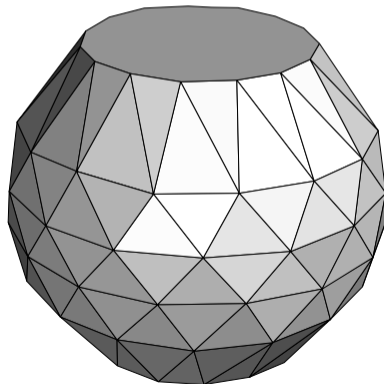
- All Platonic solids are Rupert.
- 3 Archimedean solids remain open. One of them is point-symmetric.
- Rhombicosidodecahedron natural candidate for disproving conjecture.
- *Rupertness*: Probability that a random projection yields a solution.
- Can estimate confidence intervals for Rupertness.
- Conclusion: RID is significantly different from other regular solids. It is likely that RID is a counterexample to the conjecture.



4 years later: The Noperthedron is not Rupert

Theorem (Steininger, Y., 2025)

*The Noperthedron, **NOP**, does not have Rupert's property.*



Idea of proof

- Partition the five-dimensional solution space

$$I = [0, 2\pi) \times [0, \pi] \times [0, 2\pi) \times [0, \pi] \times [-\pi, \pi)$$

into small parts and prove for each that no solution in that region exists.

- Rough idea: Show that the “middle point” of any region does not yield a solution and argue with effective continuity of the parameters that this also excludes an explicit neighborhood around that point.

Idea of proof

- Partition the five-dimensional solution space

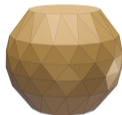
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into small parts and prove for each that no solution in that region exists.

- Rough idea: Show that the “middle point” of any region does not yield a solution and argue with effective continuity of the parameters that this also excludes an explicit neighborhood around that point.
- The **global theorem** is tailored for two generic projections of \mathbf{P} , when some vertex of the “smaller” projection $\mathcal{P} = R(\alpha)M(\theta_1, \varphi_1) \mathbf{P}$ is *strictly outside* the “larger” projection $\mathcal{Q} = M(\theta_2, \varphi_2) \mathbf{P}$.
- The **local theorem** can handle projections that look (almost) exactly the same, for instance if $\theta_1 \approx \theta_2$, $\varphi_1 \approx \varphi_2$ and $\alpha \approx 0$.

Noperthedron

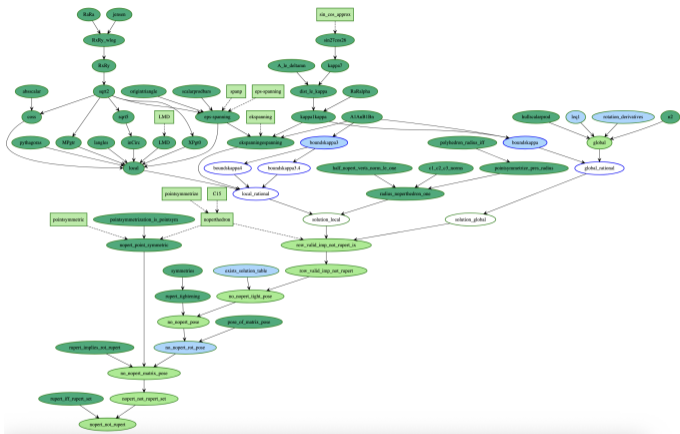
A convex polyhedron without the Rupert property

[Blueprint \(web\)](#)[Blueprint \(pdf\)](#)[Documentation](#)[GitHub](#)

This is a collaborative project to write a [Lean 4](#) proof that the Noperthedron does not have the [Rupert](#) property.

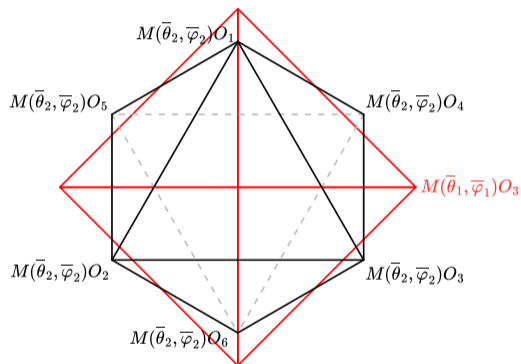
Internal links:

- [Blueprint](#)
- [Blueprint as pdf](#)
- [Dependency graph](#)
- [Doc pages for this repository](#)



Community **Project** formalizing the proof in LEAN by J. Reed and D. Renshaw.

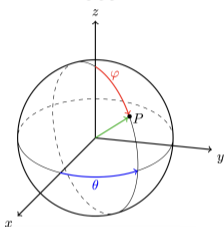
Motivation for global theorem



$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) = (0, 0, \pi/4, \tan^{-1}(\sqrt{2}), 0)$$

Starting point of the global theorem

Recall:



$$X(\theta, \varphi) := (\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)^t,$$

$$M(\theta, \varphi) := \begin{pmatrix} -\sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) \cos(\varphi) & -\sin(\theta) \cos(\varphi) & \sin(\varphi) \end{pmatrix},$$

$$R(\alpha) := \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Lemma

Let $\varepsilon > 0$ and $|\theta - \bar{\theta}|, |\varphi - \bar{\varphi}|, |\alpha - \bar{\alpha}| \leq \varepsilon$, then

- $\|X(\theta, \varphi) - X(\bar{\theta}, \bar{\varphi})\| < \sqrt{2}\varepsilon,$
- $\|M(\theta, \varphi) - M(\bar{\theta}, \bar{\varphi})\| < \sqrt{2}\varepsilon,$
- $\|R(\alpha)M(\theta, \varphi) - R(\bar{\alpha})M(\bar{\theta}, \bar{\varphi})\| < \sqrt{5}\varepsilon.$

First version of the global theorem

Recall the definition of Rupert's property

$$R(\alpha)M(\theta_1, \varphi_1) \mathbf{P} \subset (M(\theta_2, \varphi_2) \mathbf{P})^\circ.$$

If $\Psi \in \mathbb{R}^5$ is a solution, then for any vector $w \in \mathbb{R}^2$ it holds that

$$\langle R(\alpha)M(\theta_1, \varphi_1)S, w \rangle < \max_{P \in \mathbf{P}} \langle M(\theta_2, \varphi_2)P, w \rangle.$$

Theorem (Global theorem v0.1)

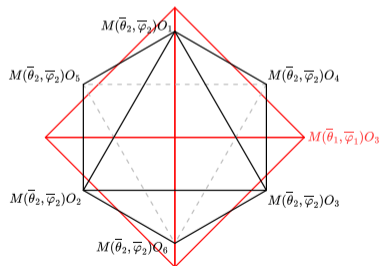
Let \mathbf{P} be convex, pointsymmetric with radius 1. Assume:

$$\langle R(\bar{\alpha})M(\bar{\theta}_1, \bar{\varphi}_1)S, w \rangle > \max_{P \in \mathbf{P}} \langle M(\bar{\theta}_2, \bar{\varphi}_2)P, w \rangle + (\sqrt{2} + \sqrt{5})\varepsilon$$

for some $S \in \mathbf{P}$ and $w \in \mathbb{R}^2$ with $\|w\| = 1$, then there cannot be a solution $(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\bar{\theta}_1 \pm \varepsilon, \bar{\varphi}_1 \pm \varepsilon, \bar{\theta}_2 \pm \varepsilon, \bar{\varphi}_2 \pm \varepsilon, \bar{\alpha} \pm \varepsilon]$.

Example

- Octahedron $\mathbf{O} = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\} \subseteq \mathbb{R}^3$ and projection directions $(\bar{\theta}_1, \bar{\varphi}_1) = (0, 0)$ and $(\bar{\theta}_2, \bar{\varphi}_2) = (\pi/4, \tan^{-1}(\sqrt{2}))$. Set $\bar{\alpha} = 0$.
- Goal: show this is not a solution to Rupert's problem and there exists $\varepsilon > 0$ such that there is also no solution $(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha)$ with $|\bar{\theta}_i - \theta_i|, |\bar{\varphi}_i - \varphi_i|, |\alpha| \leq \varepsilon$.



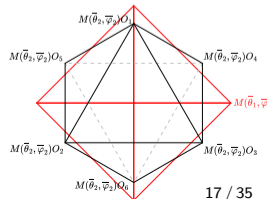
Example continued

Following the global theorem we first compute:

$$M(\bar{\theta}_1, \bar{\varphi}_1) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M(\bar{\theta}_2, \bar{\varphi}_2) = \begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{6}/6 & -\sqrt{6}/6 & \sqrt{6}/3 \end{pmatrix}.$$

We choose $S = O_3 = (0, 1, 0)$ and $w = (1, 0)$, thus $\langle R(\bar{\alpha})M(\bar{\theta}_1, \bar{\varphi}_1)S, w \rangle = 1$.

$$\langle M(\bar{\theta}_2, \bar{\varphi}_2)P, w \rangle = \begin{cases} 0 & \text{for } P = M(\bar{\theta}_2, \bar{\varphi}_2)O_i \text{ with } i = 1, 6, \\ \sqrt{2}/2 & \text{for } P = M(\bar{\theta}_2, \bar{\varphi}_2)O_i \text{ with } i = 3, 4, \\ -\sqrt{2}/2 & \text{for } P = M(\bar{\theta}_2, \bar{\varphi}_2)O_i \text{ with } i = 2, 5. \end{cases}$$



Example continued

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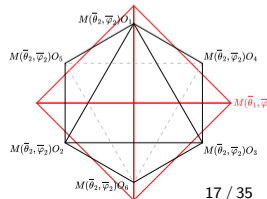
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Thus if $\varepsilon > 0$ is chosen so that

$$1 > \sqrt{2}/2 + (\sqrt{2} + \sqrt{5})\varepsilon, \quad \text{e.g., } \varepsilon = 0.08$$

there is no solution in

$$(0, 0, \pi/4, \tan^{-1}(\sqrt{2}), 0) \pm \varepsilon \subseteq \mathbb{R}^5.$$



Actual Global theorem

Theorem (Global Theorem v1.0)

Let \mathbf{P} be a pointsymmetric convex polyhedron with radius 1 and let $S \in \mathbf{P}$. Let $w \in \mathbb{R}^2$ be a unit vector and denote $\overline{M}_1 := M(\overline{\theta}_1, \overline{\varphi}_1)$, $\overline{M}_2 := M(\overline{\theta}_2, \overline{\varphi}_2)$ as well as $\overline{M}_1^\theta := M^\theta(\overline{\theta}_1, \overline{\varphi}_1)$, $\overline{M}_1^\varphi := M^\varphi(\overline{\theta}_1, \overline{\varphi}_1)$ and analogously for $\overline{M}_2^\theta, \overline{M}_2^\varphi$. Finally set

$$G := \langle R(\overline{\alpha})\overline{M}_1 S, w \rangle - \varepsilon \cdot (|\langle R'(\overline{\alpha})\overline{M}_1 S, w \rangle| + |\langle R(\overline{\alpha})\overline{M}_1^\theta S, w \rangle| + |\langle R(\overline{\alpha})\overline{M}_1^\varphi S, w \rangle|) - 9\varepsilon^2/2,$$

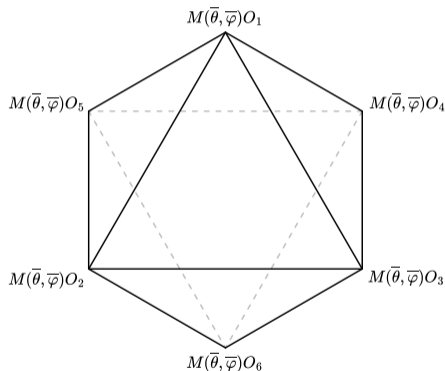
$$H_P := \langle \overline{M}_2 P, w \rangle + \varepsilon \cdot (|\langle \overline{M}_2^\theta P, w \rangle| + |\langle \overline{M}_2^\varphi P, w \rangle|) + 2\varepsilon^2, \quad \text{for } P \in \mathbf{P}.$$

If $G > \max_{P \in \mathbf{P}} H_P$ then there does not exist a solution to Rupert's condition with

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon] \subseteq \mathbb{R}^5.$$

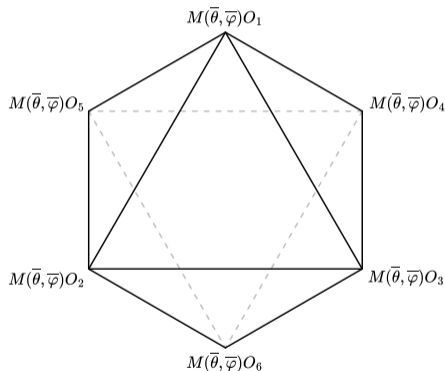
$\varepsilon = 0.08$ from the Example can be replaced by $\varepsilon = 0.164$

Motivation for local theorem: $\theta_1 = \theta_2, \varphi_1 = \varphi_2, \alpha = 0$



$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) = (\pi/4, \tan^{-1}(\sqrt{2}), \pi/4, \tan^{-1}(\sqrt{2}), 0)$$

Motivation for local theorem: $\theta_1 = \theta_2, \varphi_1 = \varphi_2, \alpha = 0$



$$A := M(\theta, \varphi)O_1,$$

$$B := M(\theta, \varphi)O_2,$$

$$C := M(\theta, \varphi)O_3$$

Fact:

$$\|A\|^2 + \|B\|^2 + \|C\|^2 = 2$$

\implies no local solution
from this direction

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) = (\pi/4, \tan^{-1}(\sqrt{2}), \pi/4, \tan^{-1}(\sqrt{2}), 0)$$

Conditions for the three-point method

- P_1, P_2, P_3 are all in front of the projection:

$$\langle X(\bar{\theta}, \bar{\varphi}), P_i \rangle > 0.$$

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Conditions for the three-point method

- P_1, P_2, P_3 are all in front of the projection:

$$\langle X(\bar{\theta}, \bar{\varphi}), P_i \rangle > \sqrt{2}\varepsilon.$$

- The origin is inside the projected triangle $M(\bar{\theta} \pm \varepsilon, \bar{\varphi} \pm \varepsilon) \cdot \{P_1, P_2, P_3\}$.

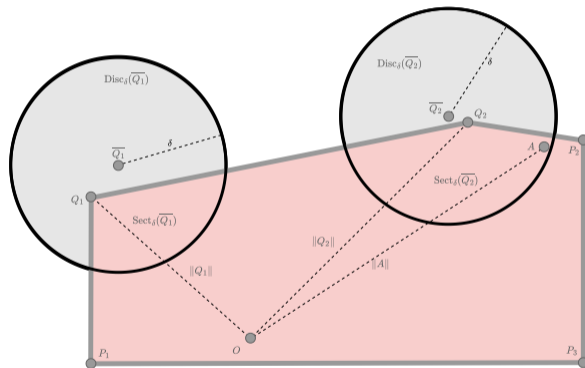
ε -spanning

- The projected points $M(\bar{\theta}, \bar{\varphi}) \cdot \{P_1, P_2, P_3\}$ are not too close to the origin:

$$\|M(\bar{\theta}, \bar{\varphi})P_i\| > r + \sqrt{2}\varepsilon.$$

- The points P_1, P_2, P_3 are ε -locally maximally distant.

Locally maximally distant points



Local Theorem

Theorem (Local Theorem v0.1)

Let \mathbf{P} be a convex, pointsymmetric polyhedron with radius 1 and $P_1, P_2, P_3 \in \mathbf{P}$. Set $\bar{X} := X(\bar{\theta}, \bar{\varphi})$, $\bar{M} := M(\bar{\theta}, \bar{\varphi})$. Assume that for all $i = 1, 2, 3$:

- $\langle \bar{X}, P_i \rangle > \sqrt{2}\varepsilon$,
- P_1, P_2, P_3 are ε -spanning for $(\bar{\theta}, \bar{\varphi})$,
- the points P_1, P_2, P_3 are ε -LMD.

Then there exists no solution to Rupert's problem with

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\bar{\theta} \pm \varepsilon, \bar{\varphi} \pm \varepsilon, \bar{\theta} \pm \varepsilon, \bar{\varphi} \pm \varepsilon, \pm\varepsilon].$$

ε -spanning points

- P_1, P_2, P_3 are all in front of the projection:

$$\langle X(\bar{\theta}, \bar{\varphi}), P_i \rangle > \sqrt{2}\varepsilon.$$

- The origin is inside the projected triangle $M(\bar{\theta} \pm \varepsilon, \bar{\varphi} \pm \varepsilon) \cdot \{P_1, P_2, P_3\}$.

ε -spanning

Lemma

Let P_1, P_2, P_3 with $\|P_1\|, \|P_2\|, \|P_3\| \leq 1$ be ε -spanning for $(\bar{\theta}, \bar{\varphi})$ and let $\theta, \varphi \in \mathbb{R}$ such that $|\theta - \bar{\theta}|, |\varphi - \bar{\varphi}| \leq \varepsilon$. Assume that $\langle X(\theta, \varphi), P_i \rangle > 0$ for $i = 1, 2, 3$. Then

$$X(\theta, \varphi) \in \text{span}^+(P_1, P_2, P_3).$$

Some more lemmas

Lemma (Pythagoras)

For any $P \in \mathbb{R}^3$ one has $\|M(\theta, \varphi)P\|^2 = \|P\|^2 - \langle X(\theta, \varphi), P \rangle^2$.

Lemma (Trinity)

Let $P_1, P_2, P_3, Y, Z \in \mathbb{R}^3$ with $\|Y\| = \|Z\|$ and $Y, Z \in \text{span}^+(P_1, P_2, P_3)$. Then at least one of the following inequalities does not hold:

$$\langle P_1, Y \rangle > \langle P_1, Z \rangle,$$

$$\langle P_2, Y \rangle > \langle P_2, Z \rangle,$$

$$\langle P_3, Y \rangle > \langle P_3, Z \rangle.$$

Proof sketch of the Local Theorem

Proof sketch.

1 Assume $(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\bar{\theta} \pm \varepsilon, \bar{\varphi} \pm \varepsilon, \bar{\theta} \pm \varepsilon, \bar{\varphi} \pm \varepsilon, \pm\varepsilon]$, let

$$M_1 = M(\theta_1, \varphi_1), M_2 = M(\theta_2, \varphi_2), X_1 = X(\theta_1, \varphi_1), X_2 = X(\theta_2, \varphi_2).$$

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- 2 Rupert's property + ε -LMD $\Rightarrow \|M_2 P_i\| > \|M_1 P_i\|$ for $i = 1, 2, 3$.
- 3 $\langle \bar{X}, P_i \rangle > \sqrt{2}\varepsilon \Rightarrow \langle X_1, P_i \rangle, \langle X_2, P_i \rangle > 0$.

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4 (2), (3) and (Pythagoras) $\Rightarrow \langle X_1, P_i \rangle > \langle X_2, P_i \rangle$

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- 5 P_1, P_2, P_3 are ε -spanning (+ Lemma) $\Rightarrow X_1, X_2 \in \text{span}^+(P_1, P_2, P_3)$

Proof sketch of the Local Theorem

Proof sketch.

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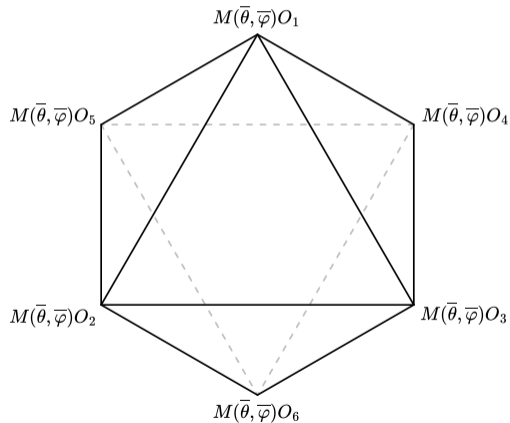
4 (2), (3) and (Pythagoras) $\Rightarrow \langle X_1, P_i \rangle > \langle X_2, P_i \rangle$

5 P_1, P_2, P_3 are ε -spanning (+ Lemma) $\Rightarrow X_1, X_2 \in \text{span}^+(P_1, P_2, P_3)$

6 (4)+(5) + Trinity lemma \Rightarrow contradiction.



Example continued



Theorem (Local Theorem v1.0)

Let \mathbf{P} be a polyhedron with radius 1 and $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in \mathbf{P}$. Assume that P_1, P_2, P_3 and Q_1, Q_2, Q_3 are congruent. Let $\varepsilon > 0$ and $\bar{\theta}_1, \bar{\varphi}_1, \bar{\theta}_2, \bar{\varphi}_2, \bar{\alpha} \in \mathbb{R}$, then set $\bar{X}_1 := X(\bar{\theta}_1, \bar{\varphi}_1)$, $\bar{X}_2 := X(\bar{\theta}_2, \bar{\varphi}_2)$ as well as $\bar{M}_1 := M(\bar{\theta}_1, \bar{\varphi}_1)$, $\bar{M}_2 := M(\bar{\theta}_2, \bar{\varphi}_2)$. Assume that there exist $\sigma_P, \sigma_Q \in \{0, 1\}$ such that

$$(-1)^{\sigma_P} \langle \bar{X}_1, P_i \rangle > \sqrt{2}\varepsilon \quad \text{and} \quad (-1)^{\sigma_Q} \langle \bar{X}_2, Q_i \rangle > \sqrt{2}\varepsilon, \quad (\text{A}_\varepsilon)$$

for all $i = 1, 2, 3$. Moreover, assume that P_1, P_2, P_3 are ε -spanning for $(\bar{\theta}_1, \bar{\varphi}_1)$ and that Q_1, Q_2, Q_3 are ε -spanning for $(\bar{\theta}_2, \bar{\varphi}_2)$. Finally, assume that for all $i = 1, 2, 3$ and any $Q_j \in \mathbf{P} \setminus Q_i$ it holds that

$$\frac{\langle \bar{M}_2 Q_i, \bar{M}_2(Q_i - Q_j) \rangle - 2\varepsilon \|Q_i - Q_j\| \cdot (\sqrt{2} + \varepsilon)}{(\|\bar{M}_2 Q_i\| + \sqrt{2}\varepsilon) \cdot (\|\bar{M}_2(Q_i - Q_j)\| + 2\sqrt{2}\varepsilon)} > \frac{\sqrt{5}\varepsilon + \delta}{r}, \quad (\text{B}_\varepsilon)$$

for some $r > 0$ such that $\min_{i=1,2,3} \|\bar{M}_2 Q_i\| > r + \sqrt{2}\varepsilon$ and for some $\delta \in \mathbb{R}$ with

$$\delta \geq \max_{i=1,2,3} \|R(\bar{\alpha})\bar{M}_1 P_i - \bar{M}_2 Q_i\| / 2.$$

Then there exists no solution to Rupert's problem $R(\alpha)M(\theta_1, \varphi_1) \mathbf{P} \subset M(\theta_2, \varphi_2) \mathbf{P}^\circ$ with $(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\bar{\theta}_1 \pm \varepsilon, \bar{\varphi}_1 \pm \varepsilon, \bar{\theta}_2 \pm \varepsilon, \bar{\varphi}_2 \pm \varepsilon, \bar{\alpha} \pm \varepsilon] \subseteq \mathbb{R}^5$.


Motivation for rational approximation

```
R 4.5.1 · ~/ ↗  
> 1 + 10^(-16) > 1  
[1] FALSE  
> |
```

```
SageMath version 10.0, Release Date: 2023-05-20  
Using Python 3.11.1. Type "help()" for help.
```

```
sage: 1 + 10^(-16) > 1  
True  
sage: █
```

Motivation for rational approximation

 R 4.5.1 · ~/ ↗

```
> 1 + 10^(-16) > 1
[1] FALSE
> |
```

imprecise but fast

SageMath version 10.0, Release Date: 2023-05-20
Using Python 3.11.1. Type "help()" for help.

```
sage: 1 + 10^(-16) > 1
True
sage: █
```

exact but slow

Idea of rational approximation $\kappa := 10^{-10}$

$$\sin_{\mathbb{Q}}(x) := x - \frac{x^3}{3} + \frac{x^5}{5!} \mp \cdots + \frac{x^{25}}{25!},$$

$$\cos_{\mathbb{Q}}(x) := 1 - \frac{x^2}{2} + \frac{x^4}{4!} \mp \cdots + \frac{x^{24}}{24!}.$$

By replacing \sin, \cos with $\sin_{\mathbb{Q}}, \cos_{\mathbb{Q}}$ define the functions

$$R_{\mathbb{Q}}(\alpha), R'_{\mathbb{Q}}(\alpha), X_{\mathbb{Q}}(\theta, \varphi), M_{\mathbb{Q}}(\theta, \varphi), M_{\mathbb{Q}}^{\theta}(\theta, \varphi), M_{\mathbb{Q}}^{\varphi}(\theta, \varphi).$$

Idea of rational approximation $\kappa := 10^{-10}$

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Lemma

Let $\alpha, \theta, \varphi \in [-4, 4]$. Then it holds that

$$\|R(\alpha) - R_{\mathbb{Q}}(\alpha)\|, \|R'(\alpha) - R'_{\mathbb{Q}}(\alpha)\|, \|X(\theta, \varphi) - X_{\mathbb{Q}}(\theta, \varphi)\|, \|M(\theta, \varphi) - M_{\mathbb{Q}}(\theta, \varphi)\| \leq \kappa.$$

Moreover,

$$\|R_{\mathbb{Q}}(\alpha)\|, \|R'_{\mathbb{Q}}(\alpha)\|, \|X_{\mathbb{Q}}(\theta, \varphi)\|, \|M_{\mathbb{Q}}(\theta, \varphi)\| \leq 1 + \kappa.$$

Rational global theorem

Theorem (Rational Global Theorem)

Let \mathbf{P} be a pointsymmetric convex polyhedron with radius $\rho = 1$ and $\tilde{\mathbf{P}}$ a κ -rational approximation. Let $\tilde{S} \in \tilde{\mathbf{P}}$. Further let $\varepsilon > 0$ and $\bar{\theta}_1, \bar{\varphi}_1, \bar{\theta}_2, \bar{\varphi}_2, \bar{\alpha} \in \mathbb{Q} \cap [-4, 4]$. Let $w \in \mathbb{Q}^2$ be a unit vector. Denote $\overline{M}_1 := M_{\mathbb{Q}}(\bar{\theta}_1, \bar{\varphi}_1)$, $\overline{M}_2 := M_{\mathbb{Q}}(\bar{\theta}_2, \bar{\varphi}_2)$ as well as $\overline{M}_1^\theta := M_{\mathbb{Q}}^\theta(\bar{\theta}_1, \bar{\varphi}_1)$, $\overline{M}_1^\varphi := M_{\mathbb{Q}}^\varphi(\bar{\theta}_1, \bar{\varphi}_1)$ and analogously for $\overline{M}_2^\theta, \overline{M}_2^\varphi$. Finally set

$$G^{\mathbb{Q}} := \langle R_{\mathbb{Q}}(\bar{\alpha})\overline{M}_1\tilde{S}, w \rangle - \varepsilon \cdot (|\langle R'_{\mathbb{Q}}(\bar{\alpha})\overline{M}_1\tilde{S}, w \rangle| + |\langle R_{\mathbb{Q}}(\bar{\alpha})\overline{M}_1^\theta\tilde{S}, w \rangle| + |\langle R_{\mathbb{Q}}(\bar{\alpha})\overline{M}_1^\varphi\tilde{S}, w \rangle|) - 9\varepsilon^2/2 - 4\kappa(1 + 3\varepsilon),$$

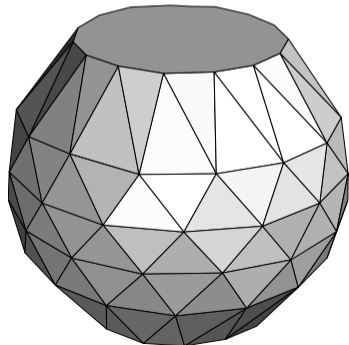
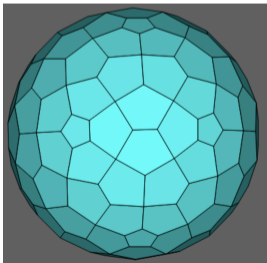
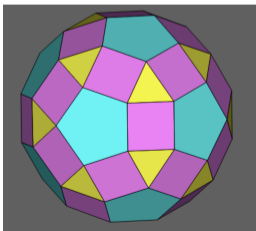
$$H_P^{\mathbb{Q}} := \langle \overline{M}_2 P, w \rangle + \varepsilon \cdot (|\langle \overline{M}_2^\theta P, w \rangle| + |\langle \overline{M}_2^\varphi P, w \rangle|) + 2\varepsilon^2 + 3\kappa(1 + 2\varepsilon).$$

If $G^{\mathbb{Q}} > \max_{P \in \tilde{\mathbf{P}}} H_P^{\mathbb{Q}}$ then there does not exist a solution to Rupert's condition with

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\bar{\theta}_1 \pm \varepsilon, \bar{\varphi}_1 \pm \varepsilon, \bar{\theta}_2 \pm \varepsilon, \bar{\varphi}_2 \pm \varepsilon, \bar{\alpha} \pm \varepsilon].$$

Wishlist for a solid

- 1 Not Rupert
- 2 many symmetries
- 3 no mirror symmetry
- 4 local theorem always applicable
- 5 point symmetry
- 6 not too many vertices



Definition of the Noperthedron

$$\mathcal{C}_{30} := \left\{ (-1)^\ell R_z \left(\frac{2\pi k}{15} \right) : k = 0, \dots, 14; \ell = 0, 1 \right\}.$$

$$C_1 := \frac{1}{259375205} \begin{pmatrix} 152024884 \\ 0 \\ 210152163 \end{pmatrix}, \quad C_2 := \frac{1}{10^{10}} \begin{pmatrix} 6632738028 \\ 6106948881 \\ 3980949609 \end{pmatrix}, \quad C_3 := \frac{1}{10^{10}} \begin{pmatrix} 8193990033 \\ 5298215096 \\ 1230614493 \end{pmatrix}.$$

Note: $\|C_1\| = 1$ and $\frac{98}{100} < \|C_i\| < \frac{99}{100}$ for $i = 2, 3$.

Definition

Define the set of points **NOP** $\subseteq \mathbb{R}^3$ by the action of \mathcal{C}_{30} on C_1, C_2, C_3 :

$$\mathbf{NOP} := \mathcal{C}_{30} \cdot C_1 \cup \mathcal{C}_{30} \cdot C_2 \cup \mathcal{C}_{30} \cdot C_3.$$

The *Noperthedron* has 90 vertices. **NOP** is pointsymmetric since $-\text{Id} \in \mathcal{C}_{30}$. Symmetries, e.g.: $M(\theta + 2\pi/15, \varphi) \cdot \mathbf{NOP} = M(\theta, \varphi) \cdot \mathbf{NOP} \Rightarrow \theta_1, \theta_2 \in [0, 2\pi/15)$.

Certificate of computer proof

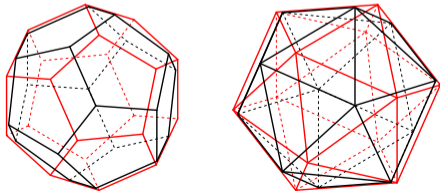
ID	T	#C	ID1C	sp	$\bar{\theta}_{1,\min}$	$\bar{\theta}_{1,\max}$	$\bar{\varphi}_{1,\min}$	$\bar{\varphi}_{1,\max}$	$\bar{\theta}_{2,\min}$	$\bar{\theta}_{2,\max}$	$\bar{\varphi}_{2,\min}$	$\bar{\varphi}_{2,\max}$	$\bar{\alpha}_{\min}$	$\bar{\alpha}_{\max}$	P_1	P_2	P_3	Q_1	Q_2	Q_3	r	σ_Q	w_x	w_y	w_d	S
0	3	4	1	1	0	64.00	0	48.00	0	64.00	0	24.00	-24.00	24.00												
1	3	30	5	2	0	16.00	0	48.00	0	64.00	0	24.00	-24.00	24.00												
2	3	30	46.67	2	16.00	32.00	0	48.00	0	64.00	0	24.00	-24.00	24.00												
3	3	30	94.77	2	32.00	48.00	0	48.00	0	64.00	0	24.00	-24.00	24.00												
4	3	30	14.51	2	48.00	64.00	0	48.00	0	64.00	0	24.00	-24.00	24.00												
5	3	4	35	3	0	16.00	0	16.00	0	64.00	0	24.00	-24.00	24.00												
6	3	4	70.31	3	0	16.00	16.00	32.00	0	64.00	0	24.00	-24.00	24.00												
7	3	4	10.67	3	0	16.00	32.00	48.00	0	64.00	0	24.00	-24.00	24.00												
...
...
91	1				0	80.00	0	80.00	80.00	16.00	80.60	16.20	-23.40	-22.80									53.73	15.64	16.45	39
92	1				0	80.00	80.00	16.00	0	80.00	0	80.60	-24.00	-23.40									98.92	35.15	10.33	37
...
245	2				0	20.00	0	20.00	0	20.00	0	20.40	-22.20	-22.80	30	31	38	79	80	87	-69	1				
246	1				0	20.00	0	20.00	0	20.00	20.40	40.80	-23.60	-22.20									71.05	19.88	20.37	39
247	1				0	20.00	0	20.00	0	20.00	20.40	40.80	-22.20	-22.80									71.05	19.88	20.37	39
248	2				0	20.00	0	20.00	20.00	40.00	0	20.40	-23.60	-22.20	30	31	38	79	80	87	-69	1				
249	2				0	20.00	0	20.00	20.00	40.00	0	20.40	-22.20	-22.80	30	31	38	79	80	87	-69	1				
...
18.44	1				48.00	64.00	46.00	48.00	48.00	64.00	22.80	24.00	22.80	24.00									33.40	-14.51	14.49	78

Computer proof statistics

- $\approx 18.000.000$ global theorem applications
- ≈ 600.000 local theorem applications
- $\approx 3\text{GB}$ uncompressed certificate ($\approx 150\text{MB}$ compressed)
- $\approx 10\text{h}$ for creation of table (using floating points in \mathbb{R})
- $\approx 30\text{h}$ for verification in SageMath

Conclusion and open question

- What about the remaining 3 Archimedean solids? In particular the Rhombicosidodecahedron?
- Is there a way to prove that a solid does not have Rupert's property without a huge case distinction? In particular, are there other ways to disprove the existence of local solutions?
- How to prove the conjectured optimal solutions?
- What is the link to duality?



Bonus: Nieuwland's number

Fact (Nieuwland, 1816)

It is possible to cut a hole inside the **unit cube** such that a **cube with side length less than** $3\sqrt{2}/4 \approx 1.0606$ can pass through this hole.

- The largest number $\nu \in \mathbb{R}$ such that $\nu\mathbf{P}$ passes through some hole inside \mathbf{P} is called *Nieuwland number of \mathbf{P}* .
- For all solids it holds that $\nu \geq 1$.
- \mathbf{P} is Rupert $\iff \nu(\mathbf{P}) > 1$.
- ν of the Cube is exactly $3\sqrt{2}/4$ [Nieuwland, 1816].

Sufficient condition of LMD

Lemma

Let \mathcal{P} be a convex polygon and $Q \in \mathcal{P}$. Let $\bar{Q} \in \mathbb{R}^2$ with $\|Q - \bar{Q}\| < \delta$ for some $\delta > 0$. Assume that for some $r > 0$ such that $\|Q\| > r$ it holds that

$$\frac{\langle Q, Q - P_j \rangle}{\|Q\| \|Q - P_j\|} \geq \frac{\delta}{r},$$

for all vertices $P_j \in \mathcal{P} \setminus Q$. Then $Q \in \mathcal{P}$ is δ -LMD(\bar{Q}).

Proof sketch.

Assume $A \in \text{Sect}_\delta(\bar{Q}) = \text{Disc}_\delta(\bar{Q}) \cap \mathcal{P}^\circ$, use $\cos(\angle(O, Q, P_j)) = \frac{\langle Q, Q - P_j \rangle}{\|Q\| \|Q - P_j\|} \geq \frac{\delta}{r}$ to conclude that $\cos(\angle(O, Q, A)) \geq \delta/r$. Therefore,

$$\|A\|^2 - \|Q\|^2 = \|Q - A\| \cdot (\|Q - A\| - 2\|Q\| \cos(\angle(O, Q, A))) < 0. \quad \square$$

Bonus: Proof of main lemma for global theorem

$$R_x(\alpha) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}, R_y(\alpha) := \begin{pmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{pmatrix}, R_z(\alpha) := \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X(\theta, \varphi)^t = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \cdot R_y(\varphi) \cdot R_z(-\theta), \quad M(\theta, \varphi) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \cdot R_y(\varphi) \cdot R_z(-\theta)$$

$$R(\alpha)M(\theta, \varphi) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \cdot R_z(\alpha) \cdot R_y(\varphi) \cdot R_z(-\theta).$$

Lemma

For any $\alpha, \beta \in \mathbb{R}$ one has $\|R_x(\alpha)R_y(\beta) - \text{Id}\| \leq \sqrt{\alpha^2 + \beta^2}$.

Sketch of proof: Write $R_x(\alpha)R_y(\beta) = UR_x(\Phi)U^{-1}$ and take trace to obtain:

$$\cos(\alpha) + \cos(\beta) + \cos(\alpha)\cos(\beta) = 1 + 2\cos(\Phi).$$

Jensen on $f(t) = \cos(\sqrt{t})$ shows $\text{LHS} \geq 1 + 2\cos(\sqrt{\alpha^2 + \beta^2})$, thus $|\Phi| \leq \sqrt{\alpha^2 + \beta^2}$.