

Integer sequences, algebraic series and differential operators¹

PhD Defense

Sergey Yurkevich

University Paris-Saclay (Inria Saclay) and
University of Vienna



universität
wien

université
PARIS-SACLAY

6th of July, 2023

¹Supervised by **Alin Bostan** and **Herwig Hauser**

Contents of the thesis I

Chapter 1: Introduction and summary of all chapters.

Chapter 2: “[On a Class of Hypergeometric Diagonals](#)”, with A. Bostan, 2022.

In: *Proceedings of the American Mathematical Society*, vol 150, pp. 1071–1897.

Chapter 3: Joint work with A. Bostan and J.-A. Weil, and: “[The art of algorithmic guessing in gfun](#)”, 2022. In: *Maple Transactions*, vol 2, pp. 14421:1–14421:19.

Chapter 4: “[A hypergeometric proof that Iso is bijective](#)”, with A. Bostan, 2022.

In: *Proceedings of the American Mathematical Society*, vol 150, pp. 2131–2136.

Chapter 5: “[Fast Computation of the \$N\$ -th Term of a \$q\$ -Holonomic Sequence and Applications](#)”, with A. Bostan, 2023. In *J. of Symbolic Comp.*, vol 115, pp. 96–123.

Contents of the thesis II

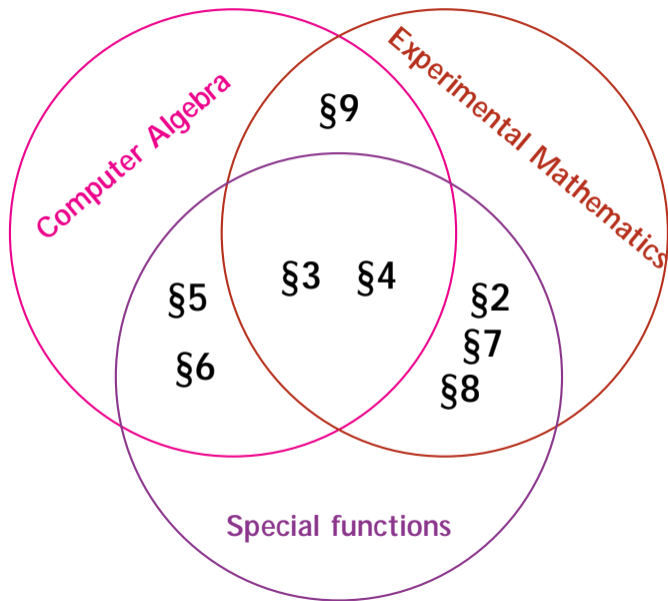
Chapter 6: “Beating binary powering for polynomial matrices”, with A. Bostan and V. Neiger, 2023. To appear in the Proceedings of *ISSAC'23*.

Chapter 7: “On the q -analogue of Pólya's Theorem”, with A. Bostan, 2023.
In: *Electronic Journal of Combinatorics*, vol 30, pp. 2.9:1-9.

Chapter 8: “On the representability of sequences as constant terms”, with A. Bostan and A. Straub, 2023. To appear in *Journal of Number Theory*.

Chapter 9: “An algorithmic approach to Rupert's problem”, with J. Steininger, 2023,
In: *Mathematics of Computation*, vol 92, pp. 1905–1929.

Chapter 10: A collection of 60 open problems and questions related to the thesis.



Chapter 2: *Hypergeometric diagonals*

$$\text{Diag}((1+x_1)^{b_1} \cdots (1+x_1 + \cdots + x_N)^{b_N}) = {}_M F_{M-1}([u]; [v]; (-N)^N t):$$

Starting point

\Guessing { that's the important beginning of solving any problem. "

- Starting point is the main identity from [Abdelaziz, Koutschan, Maillard, 2020]:

$${}_3F_2 \left(\frac{2}{9}; \frac{5}{9}; \frac{8}{9} ; 1; \frac{2}{3} ; 27t \right) = \text{Diag} \frac{(1-x-y)^{1-3}}{1-x-y-z}$$

- Left-hand side is a generalized *hypergeometric function*:

$${}_3F_2 \left(\frac{2}{9}; \frac{5}{9}; \frac{8}{9} ; 1; \frac{2}{3} ; 27t \right) := 1 + \frac{40}{9}t + \frac{5236}{81}t^2 + \dots + a_n t^n + \dots$$

- Right-hand side is the diagonal of an *algebraic function*:

$$\frac{(1-x-y)^{1-3}}{1-x-y-z} = 1 + \frac{2}{3}x + \frac{2}{3}y + z + \frac{10}{9}xy + \frac{5}{3}xz + \dots + \frac{40}{9}xyz + \dots + \frac{5236}{81}x^2y^2z^2 + \dots$$

Starting point

\Guessing { that's the important beginning of solving any problem. "

- Starting point is the main identity from [Abdelaziz, Koutschan, Maillard, 2020]:

$${}_3F_2 \left(\frac{2}{9}; \frac{5}{9}; \frac{8}{9} ; 1; \frac{2}{3} ; 27t \right) = \text{Diag} \frac{(1-x-y)^{1-3}}{1-x-y-z}$$

- Left-hand side is a generalized *hypergeometric function*:

$${}_3F_2 \left(\frac{2}{9}; \frac{5}{9}; \frac{8}{9} ; 1; \frac{2}{3} ; 27t \right) := 1 + \frac{40}{9}t + \frac{5236}{81}t^2 + \dots + a_n t^n + \dots$$

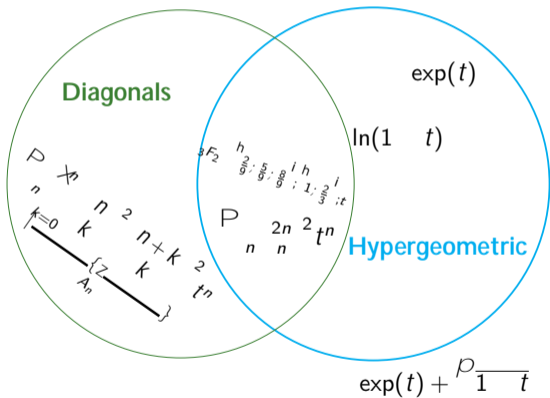
$$\frac{a_{n+1}}{a_n} = \frac{(9n+2)(9n+5)(9n+8)}{3(n+1)^2(9n+6)}$$

- Right-hand side is the diagonal of an *algebraic function*:

$$\frac{(1-x-y)^{1-3}}{1-x-y-z} = 1 + \frac{2}{3}x + \frac{2}{3}y + z + \frac{10}{9}xy + \frac{5}{3}xz + \dots + \frac{40}{9}xyz + \dots + \frac{5236}{81}x^2y^2z^2 + \dots$$

Setting

P- nite/D- nite



A sequence $(u_n)_{n \geq 0}$ is **P- nite** if it satisfies a linear recurrence with polynomial coefficients:

$$c_r(n)u_{n+r} + \dots + c_0(n)u_n = 0:$$

$(u_n)_{n \geq 0}$ is **hypergeometric** if $r = 1$.

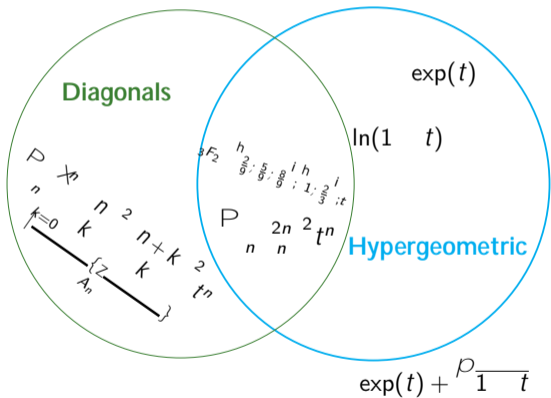
$$\text{Let } \binom{a}{n} = \frac{a!}{n!(a-n)!} \quad \binom{a+n-1}{n} = \frac{(a+n-1)!}{n!(a-1)!}:$$

Then $u_n = \frac{(a)_n (b)_n}{(c)_n n!}$ satisfies

$$(c+n)(n+1)u_{n+1} - (a+n)(b+n)u_n = 0:$$

Setting

P- nite/D- nite



A series $f(t) \in \mathbb{Q}[[t]]$ is **D- nite** if it satisfies a linear differential equation with polynomial coefficients:

$$p_r(t)f^{(r)}(t) + \dots + p_0(t)f(t) = 0:$$

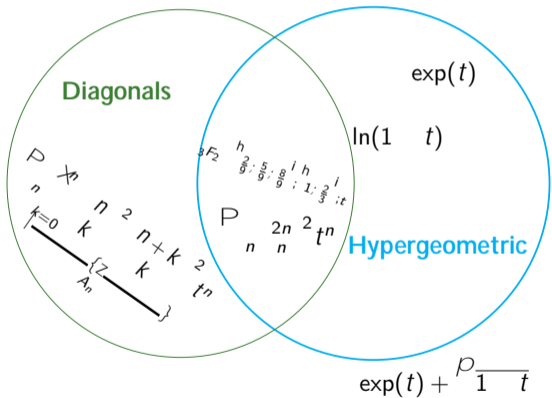
Let $(\)_n = (\ +1) (\ +n-1):$

Then ${}_2F_1 \begin{matrix} h & i \\ a & b \\ c \end{matrix}; t := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} t^n$
satisfies

$$t(1-t)f''(t) + (c - (a+b+1)t)f'(t) - abf(t) = 0:$$

Setting

P- nite/D- nite



For a multivariate power series

$$f(x_1, \dots, x_n) = \sum_{j_1, \dots, j_n} f_{j_1, \dots, j_n} x_1^{j_1} \dots x_n^{j_n}$$

the **diagonal** is given by

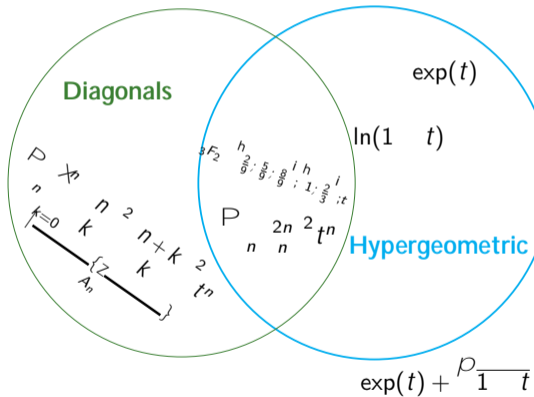
$$\text{Diag}(f) = \sum_j f_{j, j, \dots, j} t^j \in \mathbb{Q}[[t]]:$$

Diagonals are series which can be written as diagonals of **multivariate algebraic** functions.

$$\text{Diag} \frac{1}{1-xy} = \text{Diag} \sum_{i,j} \binom{i+j}{j} x^i y^j = \sum_n \binom{2n}{n} t^n = (1-4t)^{-1/2}$$

Setting

P- nite/D- nite



For a multivariate power series

$$f(x_1, \dots, x_n) = \sum_{j_1, \dots, j_n} f_{j_1, \dots, j_n} x_1^{j_1} \dots x_n^{j_n}$$

the **diagonal** is given by

$$\text{Diag}(f) = \sum_j f_{j, \dots, j} t^j \in \mathbb{Q}[[t]]:$$

Diagonals are series which can be written as diagonals of **multivariate algebraic** functions.

Christol's Conjecture [Christol, 1986]: Any convergent **D- nite** power series with integer coefficients is a **diagonal**. Specifically: ${}_3F_2 \left[\frac{1}{9}; \frac{4}{9}; \frac{5}{9}; [1; \frac{1}{3}]; t \right] \in \text{Diagonals}$.

Main result A: Hypergeometric diagonals

\First guess, then prove.

All great discoveries were made in this style."

Theorem (Bostan, Y., 2022)

The **diagonal** of any finite product of algebraic functions of the form

$$(1 - x_1 - \dots - x_n)^R; \quad R \in \mathbb{Q};$$

is a generalized **hypergeometric** function with explicitly determined parameters.

- This vastly generalizes the main identity in [Abdelaziz, Koutschan, Maillard, 2020].
- We also settle down other memberships: E.g. ${}_3F_2 \left[\frac{1}{4}; \frac{3}{8}; \frac{7}{8} \right]; \left[1; \frac{1}{3} \right]; t \in \mathbb{Z}$ **Diagonals**.
- **Main observation** for the proof:

$$\begin{aligned} & [x_1^{k_1} \dots x_N^{k_N}] (1 + x_1)^{b_1} (1 + x_1 + x_2)^{b_2} \dots (1 + x_1 + \dots + x_N)^{b_N} \\ &= \frac{b_N}{k_N} \frac{b_N - 1 + b_N}{k_N - 1} \dots \frac{b_1 + \dots + b_N - k_N}{k_1} \dots \end{aligned}$$

Chapter 3: *Dubrovin-Yang-Zagier numbers and algebraicity of D- nite functions*

$$(a_n)_{n \geq 0} = (1; 48300; 7981725900; 1469166887370000; \dots)$$

$$(b_n)_{n \geq 0} = (1; 144900; 88464128725; 62270073456990000; \dots)$$

Origin of a_n and b_n *"So this is a very mysterious example"*

- In [Arithmetic and Topology of Differential Equations, 2018](#) by Don Zagier:

$$u_n^3 + 20 \cdot 4500n^2 \cdot 18900n + 19739 \cdot u_n^2 + 80352000n(5n-1)(5n-2)(5n-4)u_n + 25 \cdot 2592000n^4 \cdot 16588800n^3 + 39118320n^2 \cdot 39189168n + 14092603 \cdot u_n - 1 = 0;$$

with initial terms $u_0 = 1; u_1 = 161 = (2^{10} \cdot 3^5)$ and $u_2 = 26605753 = (2^{23} \cdot 3^{12} \cdot 5^2)$.

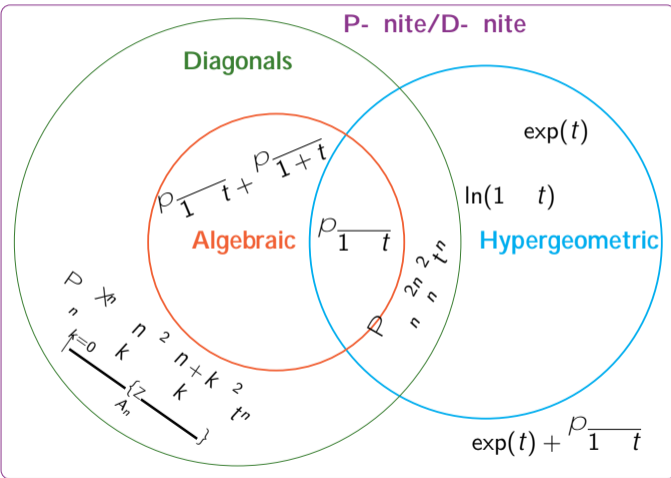
Problem (Zagier, 2018)

Find $(\alpha; \beta) \in \mathbb{Q} \times \mathbb{Q}$ such that $u_n = (\alpha)_n (\beta)_n \cdot n! \in \mathbb{Z}$ for some $n \in \mathbb{Z}$.

$$(x)_n := x(x+1)\cdots(x+n-1):$$

- [Yang and Zagier]: $a_n = u_n \cdot (3=5)_n \cdot (4=5)_n \cdot (2^{10} \cdot 3^5 \cdot 5^4)^n \in \mathbb{Z}$,
- [Dubrovin and Yang]: $b_n = u_n \cdot (2=5)_n \cdot (9=10)_n \cdot (2^{12} \cdot 3^5 \cdot 5^4)^n \in \mathbb{Z}$.
- "Yang and I found a formula showing that the numbers a_n are integers [...]"
"Dubrovin and Yang found that the numbers b_n are *also* integral and that in this case the generating function [...] is actually **algebraic!**" [Zagier, 2018]

Definitions and interactions



[Abel, 1827]: Algebraic \subseteq D- nite.

[Furstenberg, 1967]:
Algebraic \subseteq Diagonals.

[Singer 1979, 2014]:
D- nite $f(t) \stackrel{?}{\in}$ Algebraic.

[Christol, 1984 and Lipshitz, 1988]:
Diagonals \subseteq D- nite.

[Petkovsek 1992]:
D- nite $f(t) \stackrel{?}{\in}$ Hypergeometric.

[Beukers, Heckman, 1989]:
Algebraic \cap Hypergeometric.

[Bostan, Lairez, Salvy, 2017]:
Diagonals = Multiple binomial sums.

Andre-Christol Conjecture [André, 2004]:

D- nite $f(t) \in \mathbb{Z}[[t]]$ convergent & minimal ODE ordinary in 0) $f(t)$ Algebraic

Main result B: Solving the mystery of a_n and b_n

\So this is a very mysterious example."

- “Yang and I found a formula showing that the numbers a_n are integers [...]”
“Dubrovin and Yang found that the numbers b_n are *also* integral and that in this case the generating function [...] is actually **algebraic!**”
- “My presumed arithmetic intuition [...] was entirely broken” – [Wadim Zudilin]

Problem

Investigate the nature of $(a_n)_{n \geq 0}$, $(b_n)_{n \geq 0}$ and similar sequences.

Theorem (Bostan, Weil, Y.)

*The generating functions of both $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ are **algebraic**.*

Theorem (Bostan, Weil, Y.)

Seven more solutions to Zagier's problem: $(c_n)_{n \geq 0}, \dots, (i_n)_{n \geq 0} \in \mathbb{Z}$.

Chapter 4: On the reduced volume of conformal transformations of tori

Motivation and Introduction

"Why do all humans have the same biconcave shaped red blood cells?"

- Canham model predicts shape of biomembranes like blood cells [Canham, 1970].
- The model asks to minimize the Willmore energy

$$W(S) := \int_S H^2 dA; \quad (H \text{ is the mean curvature})$$

over orientable closed surfaces $S \subset \mathbb{R}^3$ with genus g , area A_0 and volume V_0 .

- [Willmore, 1965]: For a torus $T = T(R; r)$ the Willmore energy is:

$$W(T) = \frac{p}{r} \frac{2R^2}{R^2 - r^2} \quad \text{minimal for } R=r = \sqrt{\frac{p}{2}}:$$

Theorem (Willmore 1964 (conjectured); Marques, Neves, 2014)

Across all closed surfaces $S \subset \mathbb{R}^3$ of genus $g \geq 1$ the Willmore energy is minimal for $T(\sqrt{\frac{p}{2}})$.

- $W(S)$ is invariant under Möbius transformations \Rightarrow no uniqueness of the shape.

Motivation and Introduction

"Why do all humans have the same biconcave shaped red blood cells?"

- Canham model predicts shape of biomembranes like blood cells [Canham, 1970].
- The model asks to minimize the Willmore energy

$$W(S) := \int_S H^2 dA; \quad (H \text{ is the mean curvature})$$

over orientable closed surfaces $S \subset \mathbb{R}^3$ with genus g , area A_0 and volume V_0 .

- [Willmore, 1965]: For a torus $T = T(R; r)$ the Willmore energy is:

$$W(T) = \frac{p}{r} \frac{2R^2}{R^2 - r^2} \quad \text{minimal for } R=r = \sqrt{\frac{p}{2}}:$$

Theorem (Willmore 1964 (conjectured); Marques, Neves, 2014)

Across all closed surfaces $S \subset \mathbb{R}^3$ of genus $g \geq 1$ the Willmore energy is minimal for $T(\sqrt{\frac{p}{2}})$.

- $W(S)$ is invariant under Möbius transformations \Rightarrow no uniqueness of the shape.

Motivation and Introduction

"Why do all humans have the same biconcave shaped red blood cells?"

- Canham model predicts shape of biomembranes like blood cells [Canham, 1970].
- The model asks to minimize the Willmore energy

$$W(S) := \int_S H^2 dA; \quad (H \text{ is the mean curvature})$$

over orientable closed surfaces $S \subset \mathbb{R}^3$ with genus g , area A_0 and volume V_0 .

- [Willmore, 1965]: For a torus $T = T(R; r)$ the Willmore energy is:

$$W(T) = \frac{p}{r} \frac{2R^2}{R^2 - r^2} \quad \text{minimal for } R=r = \sqrt{\frac{p}{2}}:$$

Theorem (Willmore 1964 (conjectured); Marques, Neves, 2014)

Across all closed surfaces $S \subset \mathbb{R}^3$ of genus $g \geq 1$ the Willmore energy is minimal for $T(\sqrt{\frac{p}{2}})$.

- $W(S)$ is invariant under Möbius transformations \Rightarrow no uniqueness of the shape.

Main resultC: Iso is bijective

"Nature is not generic."

- In Canham's model, instead of A_0 and V_0 rather prescribe the isoperimetric ratio

$$\rho_0 := \frac{1}{A_0} \sqrt{\frac{6V_0}{\pi}} \in (0; 1]:$$

Question

Is the minimizer of $W(S)$ with prescribed genus g and isoperimetric ratio ρ_0 unique?

Theorem (Yu, Chen, 21; Melczer, Mezzarobba, 21; Bostan, Y., 22)

The shape of the projection of the Clifford torus to \mathbb{R}^3 is uniquely determined by ρ_0 . Thus, if $g = 1$ and $\rho_0 \in (0; 1]$ then Canham's model has a unique solution.

Main result C': Iso is bijective

I could never resist a definite integral."

Proposition (Bostan, Y., 2022)

The surface area $\int_0^1 \sqrt{2}^2 A(t^2)$ and volume $\int_0^1 \sqrt{2}^2 V(t^2)$ of $i_{(t;0;0)}(\mathbb{TP}^2)$ are given by

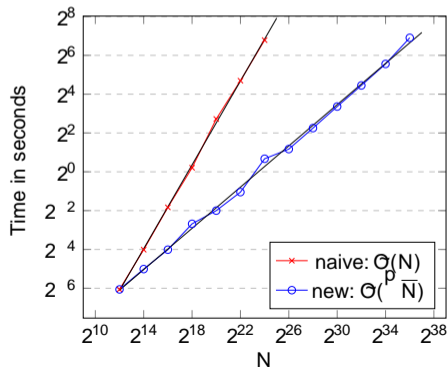
$$A(t) = \frac{4(1-t^2)}{(t^2-6t+1)^2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{4t}{(1-t)^2}\right);$$

$$V(t) = \frac{2(1-t)^3}{(t^2-6t+1)^3} {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; \frac{4t}{(1-t)^2}\right);$$

Theorem (Bostan, Y., 2022)

The function $\text{Iso}(t)^2 = 36 \frac{V(t^2)^2}{A(t^2)^3}$ is increasing on $(0; \sqrt{2}-1)$.

Chapter 5: Computing terms in q-holonomic sequences



Main resultD: Sublinear algorithm for q -holonomic sequences

- A sequence $(u_n)_{n \geq 0} \in K$ is **holonomic/P- nite** if it satisfies

$$c_r(n)u_{n+r} + \dots + c_0(n)u_n = 0 \quad n \geq 0; \quad c_0(x); \dots; c_r(x) \in K[x]:$$

Theorem (Strassen, 1977; Chudnovsky, 1988)

Given $N \in \mathbb{N}$, one can compute u_N in $\Theta(\sqrt{N})$ arithmetic operations. **Naive $O(N)$**

- A sequence $(u_n(q))_{n \geq 0} \in K$ is called **q -holonomic** if for some $q \in K$ it satisfies

$$c_r(q; q^n)u_{n+r} + \dots + c_0(q; q^n)u_n = 0 \quad n \geq 0; \quad c_0(x; y); \dots; c_r(x; y) \in K[x; y]:$$

Theorem (Bostan, Y., 2023)

Given $N \in \mathbb{N}$, one can compute $u_N(q)$ in $\Theta(\sqrt{N})$ arithmetic operations. **Naive $O(N)$**

Idea: For $M(x) \in K[x]^r$ compute $M(q^{N-1}) = M(q)M(1)$ using baby-steps/giant-steps.

Application: Evaluation of polynomials

\Do not waste a factor of two!"

- Task: Given a polynomial $P(x) \in K[x]$ and $q \in K$, deduce $P(q) \in K$ fast.
- Generically, Horner's rule needs $O(\deg P)$ operations.
- Our results imply that one can do better for large families of polynomials.
- For example, the truncated Jacobi theta function

$$\#_N(x) := 1 + x + x^4 + x^9 + \dots + x^{N^2}$$

evaluated at $q \in K$ in $\tilde{O}\left(\frac{N}{\log N}\right)$ operations [Nogneng, Schost, 2018], [Bostan, Y., 2023].

- Method: $\#_N(q) = u_N$, where $u_n = \sum_{k=0}^n q^{k^2}$ is q -holonomic.
- [Bostan, Y., 2023]: Same complexity via unified algorithm for $\sum_{i=0}^N (x^i - a^i)^Q$, or q -Hermite polynomials, or $\sum_{i=1}^N (1 - x^i)^3 \bmod x^N$, etc.

Chapter 6: Computing terms in polynomial C- nite sequences

Polynomial C- nite sequences: Example

- Fibonacci polynomials $F_0(x) = 0$; $F_1(x) = 1$ and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$
 $F_9(x) = 1 + 10x^2 + 15x^4 + 7x^6 + x^8$ and $F_{10}(x) = 5x + 20x^3 + 21x^5 + 8x^7 + x^9$:
- Compute using the definition: $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$.

- [Folklore]: Use binary powering to compute M_N , where $M_n(x) = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^n$:

$$M_n(x) = \begin{cases} M_{n/2}(x)^2 & \text{if } n \text{ even,} \\ M(x) M_{\frac{n-1}{2}}(x)^2 & \text{if } n \text{ odd.} \end{cases}$$

- Idea: Write $F_N(x) = f_0 + f_1x + \dots + f_Nx^N$. Then $(f_k)_{k=0}^N$ is P- nite :

$$f_{k+2} = \frac{(N+k+1)(N-k-1)}{4(k+1)(k+2)} f_k \quad \text{for } k \geq 0;$$

with $(f_0; f_1) = (1; 0)$ for odd N and $(f_0; f_1) = (0; N=2)$ for even N .

Polynomial C- nite sequences: Example

- Fibonacci polynomials $F_0(x) = 0$; $F_1(x) = 1$ and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$
 $F_9(x) = 1 + 10x^2 + 15x^4 + 7x^6 + x^8$ and $F_{10}(x) = 5x + 20x^3 + 21x^5 + 8x^7 + x^9$:
- Compute using the definition: $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$. $O(N^2)$

- [Folklore]: Use binary powering to compute M_N , where $M_n(x) = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^n$:

$$M_n(x) = \begin{cases} M_{n/2}(x)^2 & \text{if } n \text{ even,} \\ M(x) M_{\frac{n-1}{2}}(x)^2 & \text{if } n \text{ odd.} \end{cases}$$

- Idea: Write $F_N(x) = f_0 + f_1x + \dots + f_Nx^N$. Then $(f_k)_{k=0}^N$ is P- nite :

$$f_{k+2} = \frac{(N+k+1)(N-k-1)}{4(k+1)(k+2)} f_k \quad \text{for } k \geq 0;$$

with $(f_0; f_1) = (1; 0)$ for odd N and $(f_0; f_1) = (0; N=2)$ for even N .

Polynomial C- nite sequences: Example

- Fibonacci polynomials $F_0(x) = 0$; $F_1(x) = 1$ and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$
 $F_9(x) = 1 + 10x^2 + 15x^4 + 7x^6 + x^8$ and $F_{10}(x) = 5x + 20x^3 + 21x^5 + 8x^7 + x^9$:
- Compute using the definition: $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$. $O(N^2)$

- [Folklore]: Use binary powering to compute M_N , where $M_n(x) = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^n$:

$$M_n(x) = \begin{cases} M_{n/2}(x)^2 & \text{if } n \text{ even,} \\ M(x) M_{\frac{n-1}{2}}(x)^2 & \text{if } n \text{ odd.} \end{cases} \quad O(N \log(N))$$

- Idea: Write $F_N(x) = f_0 + f_1x + \dots + f_Nx^N$. Then $(f_k)_{k=0}^N$ is P- nite :

$$f_{k+2} = \frac{(N+k+1)(N-k-1)}{4(k+1)(k+2)} f_k \quad \text{for } k \geq 0;$$

with $(f_0; f_1) = (1; 0)$ for odd N and $(f_0; f_1) = (0; N=2)$ for even N .

Polynomial C- nite sequences: Example

- Fibonacci polynomials $F_0(x) = 0$; $F_1(x) = 1$ and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$
 $F_9(x) = 1 + 10x^2 + 15x^4 + 7x^6 + x^8$ and $F_{10}(x) = 5x + 20x^3 + 21x^5 + 8x^7 + x^9$:
- Compute using the definition: $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$. $O(N^2)$

- [Folklore]: Use binary powering to compute M_N , where $M_n(x) = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^n$:

$$M_n(x) = \begin{cases} M_{n/2}(x)^2 & \text{if } n \text{ even,} \\ M(x) M_{\frac{n-1}{2}}(x)^2 & \text{if } n \text{ odd.} \end{cases} \quad O(N \log(N))$$

- Idea: Write $F_N(x) = f_0 + f_1x + \dots + f_Nx^N$. Then $(f_k)_{k=0}^N$ is P- nite :

$$f_{k+2} = \frac{(N+k+1)(N-k-1)}{4(k+1)(k+2)} f_k \quad \text{for } k \geq 0;$$

with $(f_0; f_1) = (1; 0)$ for odd N and $(f_0; f_1) = (0; N=2)$ for even N .

$O(N)$

Main resultE: Beating binary powering \The development of fast algorithms is slow!"

A **polynomial C- nite sequence** $(u_n(x))_{n \geq 0} \in K[x]^N$ satisfies a recurrence

$$u_{n+r}(x) = c_{r-1}(x)u_{n+r-1}(x) + \dots + c_0(x)u_n(x);$$

for some polynomials $c_0(x); \dots; c_{r-1}(x) \in K[x]$.

Theorem (Bostan, Neiger, Y., 2023)

Given a **polynomial C- nite sequence** $(u_n(x))_{n \geq 0}$, one can compute $u_N(x)$ in $O(N)$ operations in K .

Corollary

Given a polynomial matrix $M(x)$, one can compute $M(x)^N$ in $O(N)$ field operations.

Chapter 7: On the q-analogue of Pólya's Theorem

Main resultF: A q-analogue of Polya's theorem

"In mathematics often the simplest is the best."

- Consequence of Polya's theorem [Polya, 1922]:

Theorem (Polya, 1922)

For admissible $n; k; a; b$, the function $F(x) := \sum_{j=0}^n \binom{n+aj}{k+bj} x^j$ is **algebraic** over $\mathbb{Q}(x)$.

- Aissen asked whether q -analogue holds [Aissen, 1979]. We prove:

Theorem (Bostan, Y., 2022)

For admissible $n; k; a; b$, the function

$$F(x; q) := \sum_{j=0}^n \binom{n+aj}{k+bj}_q x^j \in \mathbb{C}[q][[x]]$$

is never **algebraic** over $\mathbb{Q}(q; x)$. If $q \in \mathbb{C}$, then $F(x; q)$ is algebraic if q is root of unity.

- $u_n(q) = \binom{n}{k}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}$; where $[n]_q! := (1+q)(1+q+\dots+q^{n-1})$.
- Idea: It holds that $(u_n(q))_{n \geq 0}$ is **q-holonomic**.

Chapter 8: Representation of sequences as constant terms

$$\sum_{k=0}^n \binom{n}{k}^2 = \frac{(x+y)(z+1)(x+y+z)(y+x+1)}{xyz} \Big|_{x=y=z=1}^n$$

Main resultG: Describing **Constant terms** \ C- nite sequences

- A sequence $A(n)$ is a **constant term** if it can be represented as

$$A(n) = ct[P(x)^n Q(x)];$$

where $P, Q \in \mathbb{Q}[x^{-1}]$ are Laurent polynomials in $\mathbf{x} = (x_1; \dots; x_d)$.

Question (Zagier, 2018; Gorodetsky, 2021; Straub, 2022)

Which **P- nite sequences** are constant terms?

Speci cally: Are the Fibonacci numbers a constant term sequence?

Theorem (Bostan, Straub, Y., 2023)

Let $A(n)$ be a **C- nite sequence**. $A(n)$ is a **constant term** if and only if it has a single characteristic root and $\in \mathbb{Q}$.

Chapter 9: On Rupert's problem

Summary and main result: Deciding Rupertness It shows us 'what's out there'.

Definition

A convex polyhedron P in \mathbb{R}^3 is called **Rupert** if a hole with the shape of a straight tunnel can be cut into it such that a copy of P can be moved through this hole.

Theorem (Prince Rupert; Nieuwland, 1816; Scriba, 1968; Jerrard, Wetzel, Yuan, 2)

All Platonic solids are **Rupert**.

Theorem (Chai, Yuan, Zam rescu, 18; Ho mann, 18; Lavau, 19; Steininger, Y. 22)

At least 9 Archimedean solids are **Rupert**.

- [Steininger, Y., 22]: Practical algorithm and proof of algorithmic decidability.

Summary and main result: Deciding Rupertness It shows us 'what's out there'.

Definition

A convex polyhedron P in \mathbb{R}^3 is called **Rupert** if a hole with the shape of a straight tunnel can be cut into it such that a copy of P can be moved through this hole.

Theorem (Prince Rupert; Nieuwland, 1816; Scriba, 1968; Jerrard, Wetzel, Yuan, 2)

All Platonic solids are **Rupert**.

Theorem (Chai, Yuan, Zam rescu, 18; Ho mann, 18; Lavau, 19; Steininger, Y. 22)

At least 9 Archimedean solids are **Rupert**.

- [Steininger, Y., 22]: Practical algorithm and proof of algorithmic decidability.

Summary and conclusion

- A Diagonals of products of $(1 - x_1 \dots x_n)^R$ are hypergeometric functions.
- B The generating functions of the Dubrovin-Yang-Zagier numbers are algebraic.
- C Iso(t) is a quotient of hypergeometric functions and increasing. Thus the shape of a projection of the Clifford torus is uniquely determined by its isoperimetric ratio.
- D We can compute the N-th term of a q-holonomic sequence faster than previously.
- E We can compute the N-th term of a polynomial C-nite sequence faster.
- F The q-analogue of Polya's theorem holds if and only if q is a root of unity.
- G A C-nite sequence is a constant term if it has 1 characteristic root and $2 \in \mathbb{Q}$.
- H Rupertness is decidable and the truncated icosidodecahedron is Rupert.

Perspectives and open questions

"Curiouser and curiouser!"

- A? Describe **Diagonals** among **D-nite** functions.
- B? Given a **D-nite** function, how to prove or disprove that it is **algebraic** in practice?
- C? Given a **D-nite** function/**P-nite** sequence, how to prove that it is increasing?
- D? Compute N -th terms in some **P-nite** sequences faster than $O(N^P)$ operations.
- E? Compute the N -th term of an integer **C-nite sequence** in $O(N)$ bit complexity.
- F? Does there exist a suitable notion of "algebraicity"?
- G? Describe **Constant terms** among **Diagonals** or **P-nite** sequences.
- H? Prove or disprove that the Rhombicosidodecahedron is Rupert.

And many, many more...

Bonus: Definition of ${}_pF_q$ and algebraicity

The generalized **hypergeometric function** with parameters $a_1; \dots; a_p$ and $b_1; \dots; b_q$ is:

$${}_pF_q([a_1; \dots; a_p]; [b_1; \dots; b_q]; t) := \sum_{j=0}^{\infty} \frac{(a_1)_j \dots (a_p)_j}{(b_1)_j \dots (b_q)_j} \frac{t^j}{j!};$$

where $(x)_n := x(x+1)\dots(x+n-1)$ is the rising factorial.

- [Fürnsinn, Y., 2023] Can also handle the case: $a_j, b_k \notin \mathbb{Q}$ and $a_j, b_k \in \mathbb{Z}$.

Bonus: Definition of ${}_pF_q$ and algebraicity

Theorem (Christol, 1986 and Beukers, Heckman, 1989)

Assume that the rational parameters $f a_1; \dots; a_p g$ and $f b_1; \dots; b_{p-1}; b_p = 1 g$ are disjoint modulo \mathbb{Z} . Let N be their common denominator. Then

$${}_pF_{p-1}([a_1; \dots; a_p]; [b_1; \dots; b_{p-1}]; t) \quad \text{is}$$

- **algebraic** if and only if for all $1 \leq r < N$ with $\gcd(r; N) = 1$ the numbers $f \exp(2\pi i r a_j); 1 \leq j \leq p g$ and $f \exp(2\pi i r b_j); 1 \leq j \leq p-1 g$ interlace on the unit circle.
- **globally bounded** if and only if for all $1 \leq r < N$ with $\gcd(r; N) = 1$, one encounters more numbers in $f \exp(2\pi i r a_j); 1 \leq j \leq p g$ than in $f \exp(2\pi i r b_j); 1 \leq j \leq p-1 g$ when running through the unit circle from 1 to $\exp(2\pi i)$.
- [Fürnsinn, Y., 2023] Can also handle the case: $a_j; b_k \notin \mathbb{Q}$ and $a_j; b_k \in \mathbb{Z}$.

Bonus: DYZ-like numbers

Zagier's problem

Find $(;) \in \mathbb{Q} \times \mathbb{Q}$ such that $u_n (;)_n (;)_n \in \mathbb{Z}$ for some $\in \mathbb{Z}$.

$$(x)_n := x(x+1)\cdots(x+n-1):$$

#	u	v	ODE order	degree	#	u	v	ODE order	degree
a_n	3/5	4/5	2	120	f_n	19/60	49/60	4	155520
b_n	2/5	9/10	4	120	g_n	19/60	59/60	4	46080
c_n	1/5	4/5	2	120	h_n	29/60	49/60	4	46080
d_n	7/30	9/10	4	155520	i_n	29/60	59/60	4	155520
e_n	9/10	17/30	4	155520					

Theorem (Bostan, Weil, Y., 2023)

The sequences $(a_n)_{n \geq 0}; (b_n)_{n \geq 0}; (c_n)_{n \geq 0}; \dots; (i_n)_{n \geq 0}$ are solutions to Zagier's problem.

- Estimates for degrees based on numerical monodromy group computations.
- Proof of **algebraicity**: Done: $a_n; b_n; c_n$. In progress: $d_n; e_n; f_n; g_n; h_n; i_n$.