Algebraic solutions of continuous and discrete differential equations

Enumerative Combinatorics (Oberwolfach)

Sergey Yurkevich

Inria Saclay and University of Vienna

13th December, 2022

Motivating examples

- Let $F(t,u) = \sum_{n,k\geq 0} a_{n,k} t^n u^k$ be the generating function of walks in $\mathbb{N}^2$ which have n steps in $\{\nearrow, \searrow\}$ and end at level (height) k. Then:

$$F(t,u) = 1 + tuF(t,u) + t\frac{F(t,u) - F(t,0)}{u}.$$

It follows that: $F(t,0) = \dfrac{1-\sqrt{1-4t^2}}{2t^2}$. In particular, $F(t,0) \in \overline{\mathbb{Q}(t)}$.

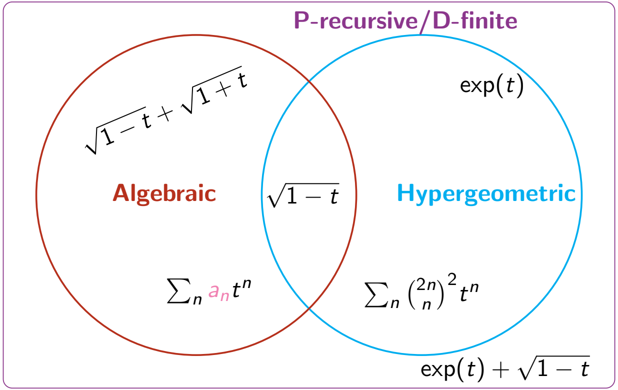
- The generating function $f(t)$ of the Yang-Zagier numbers $(a_n)_{n\geq 0}$ satisfies

$$1800t\,(7t-62)\left(t^2+50t+20\right)f''(t) + 720(42t^3+173t^2-14230t-620)f'(t) + (6048t^2-139453t-249550)f(t) = 0.$$

It follows that: $f(t) = u(t) \cdot {}_2F_1\left[\begin{matrix} -1/60 & 11/60 \\ & 2/3 \end{matrix}; q(t)\right]$, in particular, $f(t) \in \overline{\mathbb{Q}(t)}$.

[Bostan, Weil, Y., 2021]

Definitions and interactions



A power series $f(t) \in \mathbb{Q}[[t]]$ is **D-finite** if it satisfies a linear differential equation with polynomial coefficients:

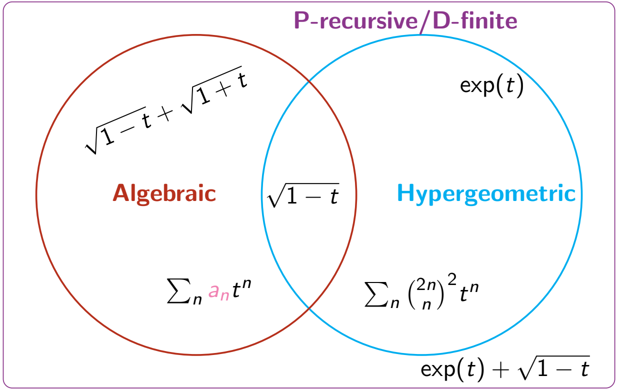
$$p_n(t)f^{(n)}(t) + \cdots + p_0(t)f(t) = 0.$$

Let $(\alpha)_n = \alpha \cdot (\alpha+1) \cdots (\alpha+n-1)$.

Then ${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; t\right] := \sum_{n\geq 0} \frac{(a)_n \cdot (b)_n}{(c)_n \cdot n!} t^n$ satisfies

$$t(1-t)f''(t) + (c-(a+b+1)t)f'(t) - abf(t) = 0.$$

Definitions and interactions



A sequence $(u_n)_{n\geq 0}$ is **P-recursive**, if it satisfies a linear recurrence with polynomial coefficients:

$$c_d(n)u_{n+d}+\cdots+c_0(n)u_n=0.$$

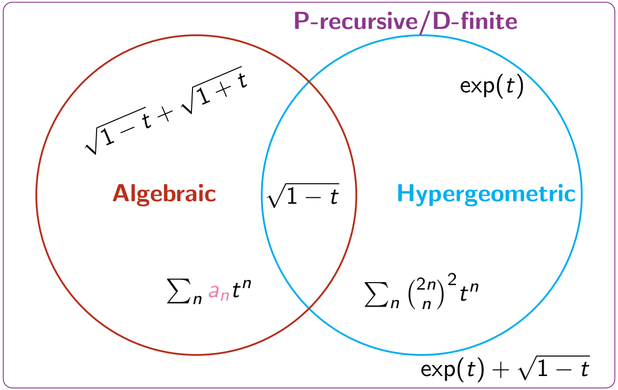
$(u_n)_{n\geq 0}$ is **hypergeometric** if $d=1$.

Let $(\alpha)_n=\alpha\cdot(\alpha+1)\cdots(\alpha+n-1)$.

Then $u_n=\frac{(a)_n\cdot(b)_n}{(c)_n\cdot n!}$ satisfies

$$(c+n)(n+1)u_{n+1}-(a+n)(b+n)u_n=0.$$

Definitions and interactions



- [Abel, 1827]:
  Algebraic $\subseteq$ **D-finite**.
- [Singer, 1979, 2014]:
  D-finite $f(t) \overset{?}{\subseteq}$ **Algebraic**.
- [Beukers, Heckman, 1989]:
  Algebraic $\cap$ **Hypergeometric**.
- [Petkovšek, 1992]:
  D-finite $f(t) \overset{?}{\subseteq}$ **Hypergeometric**.

Algebraicity of D-finite functions

Stanley's problem (1980)

Given a D-finite function how to prove (or disprove) that it is algebraic?

- Guess & Prove approach – **but** algebraicity degree can be arbitrarily high.
- Algorithms for **rational solutions** of linear ODE [Liouville, 1833; Barkatou, 1998].
- Solved **in theory** [Singer, 1979, 2014] – **but** usually not applicable in practice.
- Solved for **hypergeometric functions** [Schwarz, 1873], [Beukers,Heckman, 1989]
- Tests for justifying **algebraicity** based on **conjectures** or **numerics**:
  - **Grothendieck-Katz** conjecture (integrality of coefficients $\leftrightarrow$ algebraic solutions)
  - **Monodromy group** computation (cardinality of orbit = algebraicity degree)
- Applied differential Galois theory sometimes efficient for proving **algebraicity**.
  - Differential **Galois group** is computable [Hrushovski, 2002], [Feng, 2015].
  - Efficient computation of **Galois-Lie algebra** [Barkatou,Cluzeau,Di Vizio,Weil, 2020].

Yang-Zagier numbers $(a_n)_{n\geq 0}$

- In Arithmetic and Topology of Differential Equations, 2018 by Don Zagier:

$$c_{n-3} + 20\left(4500n^2 - 18900n + 19739\right)c_{n-2} + 80352000n(5n-1)(5n-2)(5n-4)c_n + 25\left(2592000n^4 - 16588800n^3 + 39118320n^2 - 39189168n + 14092603\right)c_{n-1} = 0,$$

with initial terms $c_0 = 1$, $c_1 = -161/(2^{10}\cdot 3^5)$ and $c_2 = 26605753/(2^{23}\cdot 3^{12}\cdot 5^2)$.

- c_n the are one-point 5-spin Witten intersection numbers [Bertola et al., 2015].

Problem (Zagier, 2018)

Find $(u,v)\in\mathbb{Q}^*\times\mathbb{Q}^*$ such that $c_n\cdot(u)_n\cdot(v)_n\cdot w^n\in\mathbb{Z}$ for some $w\in\mathbb{Z}^*$.

$$(u)_n := u\cdot(u+1)\cdots(u+n-1).$$

- [Yang and Zagier]: $a_n = c_n\cdot(3/5)_n\cdot(4/5)_n\cdot(2^{10}\cdot 3^5\cdot 5^4)^n\in\mathbb{Z}$.
- [Dubrovin and Yang]: $b_n = c_n\cdot(2/5)_n\cdot(9/10)_n\cdot(2^{12}\cdot 3^5\cdot 5^4)^n\in\mathbb{Z}$.
- [Bostan, Weil, Y.,2021]: The functions $\sum_{n\geq 0} a_n t^n$ and $\sum_{n\geq 0} b_n t^n$ are **algebraic**

… and 7 other pairs (u,v).

Discrete Differential Equations with one catalytic variable

- The divided difference operator (discrete derivative):

$$\Delta_a : \mathbb{Q}[u][[t]] \to \mathbb{Q}[u][[t]],$$
$$F(t,u) \mapsto \frac{F(t,u) - F(t,a)}{u-a}.$$

- For polynomials $f(u) \in \mathbb{Q}[u]$ and $Q \in \mathbb{Q}[x, y_1 \ldots, y_k, t, u]$ consider the equation

$$F(t,u) = f(u) + t \cdot Q(F(t,u), \Delta_a F(t,u), \ldots, \Delta_a^k F(t,u), t, u), \qquad \textbf{(DDE)}$$

where $a \in \mathbb{Q}$ (usually 0 or 1) and $k \in \mathbb{N}$ (the order of the **DDE**).

- **Algebraicity** of the (unique) solution **guaranteed** by [Popescu, 1986].
- Direct and effective proof of **algebraicity** [Bousquet-Mélou, Jehanne, 2006].
- Effective proof of **algebraicity** for **systems of DDEs** [Notarantonio, Y., 2022].

DDEs and systems of DDEs

Theorem (Bousquet-Mélou, Jehanne, 2006, Notarantonio, Y., 2022)

Let $n, k \geq 1$ be integers and $f_1, \ldots, f_n \in \mathbb{Q}[u]$, $Q_1, \ldots, Q_n \in \mathbb{Q}[y_1, \ldots, y_{n(k+1)}, t, u]$ be polynomials. Set $\nabla^k F := F, \Delta_a F, \ldots, \Delta_a^k F$. Then the system of equations

$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \ldots, \nabla^k F_n, t, u), \\ \quad \vdots \qquad\qquad\qquad \vdots \\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \ldots, \nabla^k F_n, t, u) \end{cases}$$

admits a unique vector of solutions $(F_1, \ldots, F_n) \in \mathbb{Q}[u][[t]]^n$, and all F_i are algebraic functions over $\mathbb{Q}(t, u)$. Moreover, there exists an algorithm for computing the minimal polynomial of each $F_i(t, u)$.

DDEs and systems of DDEs

Theorem (Bousquet-Mélou, Jehanne, 2006, Notarantonio, Y., 2022)

A system of DDEs *admits a unique vector of solutions* $(F_1, \dots, F_n) \in \mathbb{Q}[u][[t]]^n$*, and all* F_i *are algebraic functions over* $\mathbb{Q}(t,u)$*. Moreover, there exists an algorithm for computing the minimal polynomial of each* $F_i(t,u)$*.*

Example (introduced and solved in [Bonichon,Bousquet-Mélou,Dorbec,Pennarun, 2006]):

$$\begin{cases} F_1(t,u) = 1 + t\cdot\left(u + 2uF_1(t,u)^2 + 2uF_2(t,1) + u\frac{F_1(t,u)-uF_1(t,1)}{u-1}\right), \\ F_2(t,u) = t\cdot\left(2uF_1(t,u)F_2(t,u) + uF_1(t,u) + uF_2(t,1) + u\frac{F_2(t,u)-uF_2(t,1)}{u-1}\right). \end{cases}$$

$G = F_1(t,0)$ and $F_2(t,0)$ are algebraic functions. For example:

$$64t^3G^3 + 2t\left(24t^2 - 36t + 1\right)G^2 - \left(15t^3 - 9t^2 - 19t + 1\right)G + t^3 + 27t^2 - 19t + 1 = 0.$$

Conclusion

- Proving **algebraicity** of solutions of ODEs can be difficult.
  - Guess & Prove approach often useful but sometimes infeasible.
  - Heuristic methods (**Grothendieck-Katz** conjecture, numerical **monodromy group** calculations) allow efficient "testing".
  - Effective **differential Galois theory** can be used for proving.

- The generating function of the Yang-Zagier numbers a_n is **algebraic**.

- **Systems of DDEs** with one catalytic variable **always** have **algebraic** solutions.

- There exists an algorithm for finding minimal polynomials of such solutions

  ... and ongoing work on efficiency and more catalytic variables.

Bonus: Yang-Zagier numbers $(a_n)_{n\geq 0}$ and more

Problem

Find $(u, v) \in \mathbb{Q}^* \times \mathbb{Q}^*$ such that $c_n \cdot (u)_n \cdot (v)_n \cdot w^n \in \mathbb{Z}$ for some $w \in \mathbb{Z}^*$.

$$(u)_n := u \cdot (u+1) \cdots (u+n-1).$$

#	u	v	ODE order	degree	#	u	v	ODE order	degree
a_n	$3/5$	$4/5$	2	120	f_n	$19/60$	$49/60$	4	155520
b_n	$2/5$	$9/10$	4	120	g_n	$19/60$	$59/60$	4	46080
c_n	$1/5$	$4/5$	2	120	h_n	$29/60$	$49/60$	4	46080
d_n	$7/30$	$9/10$	4	155520	i_n	$29/60$	$59/60$	4	155520
e_n	$9/10$	$17/30$	4	155520					

- "Test": 0 p-curvatures for primes < 100 $\rightarrow$ expect **algebraic** generating functions.
- Quantify: Guesses for degrees based on numerics.
- Proof: Done: a_n, b_n, c_n, g_n. In progress: d_n, e_n, f_n, h_n, i_n.

Bonus: Transcendence of D-finite functions

Stanley's problem (1980)

Given a **D-finite** function how to disprove that it is **algebraic**?

Some useful properties of algebraic functions $f(t) = \sum_{n\geq 0} u_n t^n$:

- Coefficient sequence is globally bounded — $\log(1-t) \notin \overline{\mathbb{Q}(t)}$
- Special asymptotics: $u_n = \frac{\rho^n n^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^n C_i \omega_i^n + O(\rho^n n^\beta)$ — $\sum_{n\geq 0} \binom{2n}{n}^2 t^n \notin \overline{\mathbb{Q}(t)}$
- Evaluation at algebraic numbers: $f(\alpha) \in \overline{\mathbb{Q}}$ for $\alpha \in \overline{\mathbb{Q}}$. — $\exp(t) \notin \overline{\mathbb{Q}(t)}$
- Minimal differential operator has logarithmic singularity

$$\sum_{n\geq 0} \sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 t^n \notin \overline{\mathbb{Q}(t)}$$

$$\sum_{n\geq 0} u_n t^n \notin \overline{\mathbb{Q}(t)},\ \text{where } u_n \text{ counts walks in } \mathbb{N}^2 \text{ with steps in } \{\nwarrow, \nearrow, \downarrow\}$$

Bonus: Algorithm for systems of DDEs

$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \ldots, \nabla^k F_n, t, u), \\ \vdots \qquad\qquad\qquad \vdots \\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \ldots, \nabla^k F_n, t, u) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} G_1 = f_1(u) + t^\alpha \cdot Q_1(\nabla^k G_1, \nabla^k G_2, \ldots, \nabla^k G_n, t^\alpha, u) + t \cdot \epsilon^k \cdot \sum_{i=1}^n \gamma_{1,i} \cdot \Delta^k G_i, \\ \vdots \qquad\qquad\qquad \vdots \\ G_n = f_n(u) + t^\alpha \cdot Q_n(\nabla^k G_1, \nabla^k G_2, \ldots, \nabla^k G_n, t^\alpha, u) + t \cdot \epsilon^k \cdot \sum_{i=1}^n \gamma_{n,i} \cdot \Delta^k G_i, \end{cases}$$

1. Define $E_1, \ldots, E_n \in \mathbb{Q}[x, z_1, \ldots, z_{nk}, t, u, \epsilon]$ as the numerators of $\mathbf{DDE}_\epsilon$.
2. Compute $\mathrm{Det}, \mathrm{P} \in \mathbb{Q}[x_1, \ldots, x_n, u_1, \ldots, u_{nk}, z_1, \ldots, z_{nk}]$.
3. Define the system $\mathcal{S}_{\mathrm{dup}}$ (in $nk(n+2)$ equations and variables).
4. Saturate $\mathcal{S}_{\mathrm{dup}}$ by adding the equation $m \cdot \det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}}) - 1 = 0$ for a variable m.
5. Compute a non-zero element of $\mathcal{S}_{\mathrm{sat}} \cap \mathbb{Q}[z_0, t]$.