

# Algebraicity of solutions of DDEs with one catalytic variable<sup>1</sup>

MathExp (Inria Saclay)

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<sup>1</sup>Joint work with [Hadrien Notarantonio](#). Accepted as a talk at [FPSAC23](#).

## Motivating examples

- Let  $F(t, u) = \sum_{n, k \geq 0} a_{n, k} t^n u^k$  be the generating function of walks in  $\mathbb{N}^2$  which have  $n$  steps in  $\{\nearrow, \searrow\}$  and end at level (height)  $k$ . One finds:

$$F(t, u) = 1 + tuF(t, u) + t \frac{F(t, u) - F(t, 0)}{u}.$$

It follows that:  $F(t, 0) = \frac{1 - \sqrt{1 - 4t}}{2t}$ . In particular:  $F(t, 0) \in \overline{\mathbb{Q}(t)}$ .

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- Modelling [Bonichon, Bousquet-Mélou, Dorbec, Pennarun, 2006] special Eulerian planar orientations gives rise to:

$$\begin{cases} F_1(t, u) = 1 + t \cdot \left( u + 2uF_1(t, u)^2 + 2uF_2(t, 1) + u \frac{F_1(t, u) - uF_1(t, 1)}{u-1} \right), \\ F_2(t, u) = t \cdot \left( 2uF_1(t, u)F_2(t, u) + uF_1(t, u) + uF_2(t, 1) + u \frac{F_2(t, u) - uF_2(t, 1)}{u-1} \right). \end{cases}$$

Again:  $G = F_1(t, 0)$  and  $F_2(t, 0)$  are algebraic functions, for example:

$$64t^3 G^3 + 2t(24t^2 - 36t + 1)G^2 - (15t^3 - 9t^2 - 19t + 1)G + t^3 + 27t^2 - 19t + 1 = 0.$$

# Discrete Differential Equations (DDEs) with one catalytic variable

- The divided difference operator (discrete derivative):

$$\Delta_a : \mathbb{Q}[u][[t]] \rightarrow \mathbb{Q}[u][[t]],$$
$$F(t, u) \mapsto \frac{F(t, u) - F(t, a)}{u - a}.$$

- $\Delta_a^j$  is the  $j$ -th iteration of  $\Delta_a$ . Explicitly:

$$\Delta_a^{j+1} F(t, u) = \frac{F(t, u) - F(t, a) - (u - a)\partial_u F(t, a) - \dots - \frac{(u-a)^j}{j!} \partial_u^j F(t, a)}{(u - a)^{j+1}}$$

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- For polynomials  $f(u) \in \mathbb{Q}[u]$  and  $Q \in \mathbb{Q}[x, y_1, \dots, y_k, t, u]$  consider the equation

$$F(t, u) = f(u) + t \cdot Q(F(t, u), \Delta_a F(t, u), \dots, \Delta_a^k F(t, u), t, u), \quad \text{(DDE)}$$

where  $a \in \mathbb{Q}$  (usually 0 or 1) and  $k \in \mathbb{N}$  (the order of the **DDE**).

## Some history

- In 1960s, Tutte reduced many combinatorial problems to studying **DDEs**.  
E.g.: [Tutte, 1962] and [Brown, Tutte 1964]
- In 1986 Popescu proved the “General Néron desingularization” [Popescu, 1986].
- [Bandarier, Flajolet 2002]: Universal “Kernel method” for linear **DDEs**.
- In 2006: The unique solution of any **DDE** is an algebraic function and effective method to compute the minimal polynomial [Bousquet-Mélou, Jehanne, 2006].
- In 2015 [Hauser, Rond] organize a conference on “Artin Approximation”:  
Popescu’s Theorem also implies algebraicity of **DDEs** (but in a non-effective way).
- [Buchacher, Kauers, 2020]: Linear systems of **DDEs** have algebraic solutions  
(effective proof).
- [Bostan, Chyzak, Notarantonio, Safey El Din, 2022]: Fast algorithms for order 1.
- **New:** [Notarantonio, Y., 2022]: Effective proof that systems of **DDEs** have algebraic solutions.

# Main result for scalar equations

## Theorem (Bousquet-Mélou, Jehanne, 2006)

Let  $k \geq 1$ ,  $a \in \mathbb{Q}$  and  $Q \in \mathbb{Q}[x, y_1, \dots, y_k, t, u]$ ,  $f(u) \in \mathbb{Q}[u]$ . There exists a unique solution  $F(t, u) \in \mathbb{Q}[u][[t]]$  of the functional equation

$$F(t, u) = f(u) + t \cdot Q(F(t, u), \Delta_a F(t, u), \dots, \Delta_a^k F(t, u), t, u),$$

and  $F(t, u)$  is algebraic over  $\mathbb{Q}(t, u)$ . Moreover, there exists an algorithm for computing the minimal polynomial of  $F(t, u)$ .

## Sketch of proof (generic case)

- Let  $E$  be the polynomial in  $x, z_0, \dots, z_{k-1}, t, u$  that is induced by **DDE**:

$$E(u) := E(\underbrace{F(t, u)}_x, \underbrace{F(t, 0)}_{z_0}, \dots, \underbrace{\partial_u^{k-1} F(t, 0)}_{z_{k-1}}, t, u) = 0.$$



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- Take the derivative of  $E(u) = 0$  with respect to  $u$ :

$$\partial_u F(t, u) \cdot \partial_x E(u) + \partial_u E(u) = 0. \quad (\partial_x E(u) \text{ is the "kernel"})$$

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- Any solution  $u = U(t)$  of  $\partial_x E(u) = 0$  also implies  $\partial_u E(u) = 0$ .
- Obtain 3 equations ( $E(u) = 0, \partial_x E(u) = 0, \partial_u E(u) = 0$ ) for  $k + 1 + 1$  variables.

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- If  $\partial_x E(u) = 0$  has  $k$  distinct solutions  $U_1, \dots, U_k$ , we can consider:

$$\mathcal{S}_{\text{dup}} := \begin{cases} E(u_i) = 0, \\ \partial_x E(u_i) = 0, \\ \partial_u E(u_i) = 0. \end{cases} \quad \text{for } i = 1, \dots, k. \quad \begin{cases} F(t, U_i) \leftrightarrow x_i, \\ \partial_u^i F(t, 0) \leftrightarrow z_i, \\ U_i \leftrightarrow u_i. \end{cases}$$

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- We find:  $3k$  equations and  $3k$  variables. Hope: the system  $\mathcal{S}_{\text{dup}}$  is 0-dimensional.
- In that case can use elimination algorithms and find the annihilating polynomial.

# Sketch of proof

- Two issues:
  - 1  $\partial_x E(u) = 0$  does not always have  $k$  distinct solutions.
  - 2 Is  $\mathcal{S}_{\text{dup}}$  really 0-dimensional?

# Sketch of proof

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- $\partial_x E(u) = 0$  does not always have  $k$  distinct solutions.

- Is  $\mathcal{S}_{\text{dup}}$  really 0-dimensional?

- Given the **DDE**

$$F(t, u) = f(u) + t \cdot Q(F(t, u), \Delta_a F(t, u), \dots, \Delta_a^k F(t, u), t, u),$$

consider the perturbed **DDE** $_{\epsilon}$ :

$$G(t, u, \epsilon) = f(u) + t^2 \cdot Q(G(t, u, \epsilon), \Delta_a G(t, u, \epsilon), \dots, \Delta_a^k G(t, u, \epsilon), t^2, u) + t\epsilon^k \Delta^k G(t, u, \epsilon).$$

- $G(t, u, \epsilon)$  algebraic over  $\mathbb{Q}(t, u, \epsilon) \Rightarrow F(t, u)$  algebraic over  $\mathbb{Q}(t, u)$ .

## Lemma 1

Let  $E_G$  be the numerator of  $\text{DDE}_{\epsilon}$ .  $\partial_x E_G(u) = 0$  has  $k$  distinct solutions in  $\overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{k}}]]$ .

## Lemma 2

The ideal  $\langle \mathcal{S}_{\text{dup}} \rangle : \det(\text{Jac}_{\mathcal{S}_{\text{dup}}})^{\infty}$  is 0-dimensional and contains the minimal poly. of  $G$ .

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- The equation  $\partial_x E_G = 0$  has the form:

$$u^k = \epsilon^k t + t^2 \sum_{i=0}^k u^{k-i} \partial_{y_i} Q(F(t, u), \dots, \Delta^k F(t, u), t^2, u)$$

- Newton's algorithm implies that we find  $k$  solutions of the form

$$U_\ell(t) = \epsilon \cdot t^{\frac{1}{k}} \cdot \zeta^\ell + O(t^{\frac{2}{k}}),$$

for  $\zeta$  a primitive  $k$ -th root of unity.



# Sketch of proof (Lemma 2)

## Lemma 2

The ideal  $\langle \mathcal{S}_{\text{dup}} \rangle : \det(\text{Jac}_{\mathcal{S}_{\text{dup}}})^\infty$  is 0-dimensional and contains the minimal poly. of  $G$ .

- Recall:  $E_G(u) := E_G(F(t, u), F(t, 0), \dots, \partial_u^{k-1} F(t, 0), t, u)$  where  $E_G$  describes  $G(t, u, \epsilon) = f(u) + t^2 \cdot Q(G(t, u, \epsilon), \Delta_a G(t, u, \epsilon), \dots, \Delta_a^k G(t, u, \epsilon), t^2, u) + t\epsilon^k \Delta^k G(t, u, \epsilon)$ .

$$\mathcal{S}_{\text{dup}} := \{E_G(u_i) = 0, \partial_x E_G(u_i) = 0, \partial_u E_G(u_i) = 0 : i = 1, \dots, k\}.$$

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- Application of Hilbert's Nullstellensatz: Lemma 2 holds if the Jacobian matrix  $\text{Jac}_{\mathcal{S}_{\text{dup}}} \in \overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{*}}]]^{3k \times 3k}$  of  $\mathcal{S}_{\text{dup}}$  is invertible at

$$(x_1, \dots, x_k, u_1, \dots, u_k, z_0, \dots, z_{k-1}) =$$

$$(G(t, U_1), \dots, G(t, U_k), U_1(t), \dots, U_k(t), G(t, 0), \dots, \partial_u^{k-1} G(t, 0)) \in \overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{*}}]]^{3k}.$$

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- Bousquet-Mélou and Jehanne compute the determinant explicitly:

$$\det(\text{Jac}_{\mathcal{S}_{\text{dup}}}) = r \cdot \prod_{i < j} (\zeta^i - \zeta^j) \cdot \prod_{i=0}^k (\partial_x^2 E(U_j) \partial_u^2 E(U_j) - \partial_x \partial_u E(U_j)^2) + O(t^k) \neq 0.$$

# Summary of the proof and algorithm in the scalar case

## ■ Strategy of the proof:

- 1 Given a **DDE** for  $F(t, u)$  consider the perturbed **DDE** $_{\epsilon}$ :

$$G(t, u, \epsilon) = f(u) + t^2 \cdot Q(G(t, u, \epsilon), \Delta_a G(t, u, \epsilon), \dots, \Delta_a^k G(t, u, \epsilon), t^2, u) + t\epsilon^k \Delta^k G(t, u, \epsilon).$$

and let  $E_G$  be the defining polynomial in the variables  $x, z_0, \dots, z_{k-1}, t, u, \epsilon$ .

- 2 Prove that  $\partial_x E_G(u) = 0$  has  $k$  distinct solutions  $u = U(t)$  in  $\overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{k}}]]$ .
- 3 Define  $\mathcal{S} = (E, \partial_x E_G, \partial_u E_G)$  and let  $\mathcal{S}_{\text{dup}}$  be the duplicated system.
- 4 Show that  $\langle \mathcal{S}_{\text{dup}} \rangle : \det(\text{Jac}_{\mathcal{S}_{\text{dup}}})^{\infty}$  is 0-dimensional by proving:  $\text{Jac}_{\mathcal{S}_{\text{dup}}}$  is invertible.

## ■ Algorithm:

- 1 Define  $E_G \in \mathbb{Q}[x, z_0, \dots, z_{k-1}, t, u, \epsilon]$  as the numerator of of **DDE** $_{\epsilon}$ .
- 2 Compute  $\partial_x E$  and  $\partial_u E$ . Define  $\mathcal{S}_{\text{dup}}$  in  $x_1, \dots, x_n, u_1, \dots, u_n, z_0, \dots, z_{k-1}$ .
- 3 Saturate  $\mathcal{S}_{\text{dup}}$  by adding the equation  $m \cdot \det(\text{Jac}_{\mathcal{S}_{\text{dup}}}) - 1 = 0$  for a variable  $m$ .
- 4 Compute a non-zero element of  $\mathcal{S}_{\text{sat}} \cap \mathbb{Q}[z_0, t]$ .

# New result: extension for systems

## Theorem (Notarantonio, Y., 2022)

Let  $n, k \geq 1$  be integers and  $f_1, \dots, f_n \in \mathbb{Q}[u]$ ,  $Q_1, \dots, Q_n \in \mathbb{Q}[y_1, \dots, y_{n(k+1)}, t, u]$  be polynomials. Set  $\nabla^k F := F, \Delta_a F, \dots, \Delta_a^k F$ . Then the system of equations

$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \dots, \nabla^k F_n, t, u), \\ \vdots \\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \dots, \nabla^k F_n, t, u) \end{cases}$$

admits a unique vector of solutions  $(F_1, \dots, F_n) \in \mathbb{Q}[u][[t]]^n$ , and all its components are algebraic functions over  $\mathbb{Q}(t, u)$ .

# Example

## Theorem (Notarantonio, Y., 2022)

*A system of DDEs with one catalytic variable admits a unique vector of solutions  $(F_1, \dots, F_n) \in \mathbb{Q}[u][[t]]^n$ , and all its components are algebraic functions over  $\mathbb{Q}(t, u)$ .*

Example (introduced and solved in [\[Bonichon, Bousquet-Mélou, Dorbec, Pennarun, 2006\]](#)):

$$\begin{cases} F_1(t, u) = 1 + t \cdot \left( u + 2uF_1(t, u)^2 + 2uF_2(t, 1) + u \frac{F_1(t, u) - uF_1(t, 1)}{u-1} \right), \\ F_2(t, u) = t \cdot \left( 2uF_1(t, u)F_2(t, u) + uF_1(t, u) + uF_2(t, 1) + u \frac{F_2(t, u) - uF_2(t, 1)}{u-1} \right). \end{cases}$$

It holds:  $G = F_1(t, 0)$  and  $F_2(t, 0)$  are algebraic functions. For example:

$$64t^3 G^3 + 2t(24t^2 - 36t + 1)G^2 - (15t^3 - 9t^2 - 19t + 1)G + t^3 + 27t^2 - 19t + 1 = 0.$$

## Generic system case

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defines  $n$  polynomial equations  $E_1 = 0, \dots, E_n = 0$  in  $\mathbb{Q}[x_1, \dots, x_n, z_0, \dots, z_{nk-1}, u, t]$ .

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- The “derivative” of  $(E_1, \dots, E_n)$  with respect to  $u$ :

$$\begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_n} E_n \end{pmatrix} \cdot \begin{pmatrix} \partial_u F_1 \\ \vdots \\ \partial_u F_n \end{pmatrix} + \begin{pmatrix} \partial_u E_1 \\ \vdots \\ \partial_u E_n \end{pmatrix} = 0. \quad (1)$$



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- Now define:

$$\text{Det} := \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_n} E_n \end{pmatrix} \quad \text{and} \quad P := \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_1} E_{n-1} & \partial_{x_1} E_n \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_{n-1}} E_1 & \dots & \partial_{x_{n-1}} E_{n-1} & \partial_{x_{n-1}} E_n \\ \partial_u E_1 & \dots & \partial_u E_{n-1} & \partial_u E_n \end{pmatrix}.$$

- Observation: (1) and  $\text{Det} = 0$  imply  $P = 0$ .

Therefore we have:  $n + 2$  **equations**, and  $n + nk + 1$  **variables**.

## Generic system case

- Equations:  $\mathcal{S} = \{E_1(u), \dots, E_n(u), \text{Det}(u), P(u)\}$ .
- If  $\text{Det}(u) = 0$  has  $nk$  distinct solutions in  $\overline{\mathbb{Q}}[[t^{\frac{1}{*}}]]$ , then define the system:

$$\mathcal{S}_{\text{dup}} := \begin{cases} E_1(u_i) = E_2(u_i) = \dots = E_n(u_i) = 0, \\ \text{Det}(u_i) = 0, \\ P(u_i) = 0. \end{cases} \quad \text{for } i = 1, \dots, nk.$$

- In total  $(n + 2) \cdot nk = nk(n + 2)$  equations.
- Variables:  $\underbrace{x_1, \dots, x_{n^2k}}_{F_i(U_j)}, \underbrace{z_0, \dots, z_{nk-1}}_{\partial^j F_i(t,a)}, \underbrace{u_1, \dots, u_{nk}}_{U_i} \Rightarrow n^2k + nk + nk = nk(n + 2)$ .
- $\Rightarrow$  Can hope for a 0-dimensional system.

# Outline of the proof for systems in the general case

- 1 Given a system of **DDEs** for  $F_i(t, u)$  consider the perturbed system **DDE** $_{\epsilon}$ .
- 2 Prove that  $\text{Det}(u) = 0$  has  $nk$  distinct solutions  $u = U(t)$  in  $\overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{\star}}]]$ .
- 3 Define  $\mathcal{S} = (E_1, \dots, E_n, \text{Det}, P)$  and let  $\mathcal{S}_{\text{dup}}$  be the duplicated system in  $nk(n+2)$  variables.
- 4 Show:  $\langle \mathcal{S}_{\text{dup}} \rangle : \det(\text{Jac}_{\mathcal{S}_{\text{dup}}})^{\infty}$  is 0-dimensional by proving:  $\text{Jac}_{\mathcal{S}_{\text{dup}}}$  is invertible.

## Sketch of proof: Step 1, deformation

$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \dots, \nabla^k F_n, t, u), \\ \vdots \\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \dots, \nabla^k F_n, t, u) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} G_1 = f_1(u) + t^\alpha \cdot Q_1(\nabla^k G_1, \nabla^k G_2, \dots, \nabla^k G_n, t^\alpha, u) + t \cdot \epsilon^k \cdot \sum_{i=1}^n \gamma_{1,i} \cdot \Delta^k G_i, \\ \vdots \\ G_n = f_n(u) + t^\alpha \cdot Q_n(\nabla^k G_1, \nabla^k G_2, \dots, \nabla^k G_n, t^\alpha, u) + t \cdot \epsilon^k \cdot \sum_{i=1}^n \gamma_{n,i} \cdot \Delta^k G_i, \end{cases}$$

where  $\gamma_{i,i} = i^k$  and  $\gamma_{i,j} = t^\beta$ , and  $\alpha \gg \beta \gg 0$  are chosen sufficiently large.

- It holds:  $G_i(t, u, \epsilon)$  algebraic over  $\mathbb{Q}(t, u, \epsilon) \Rightarrow F_i(t, u)$  algebraic over  $\mathbb{Q}(t, u)$ .
- Each equation for  $G_i$  induces a polynomial equation:

$$E_i(\nabla^k G_1, \nabla^k G_2, \dots, \nabla^k G_n, t, u, \epsilon) = 0, \quad i = 1, \dots, n.$$

## Step 2: Definition of $\text{Det}(u)$ and proof of $nk$ distinct solutions

$$\text{Det} := \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_n} E_n \end{pmatrix}$$

### Lemma 1

The equation  $\text{Det}(u) = 0$  has  $nk$  distinct solutions in  $\overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{k}}]]$

## Step 2: Definition of $\text{Det}(u)$ and proof of $nk$ distinct solutions

$$\text{Det} := \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_n} E_n \end{pmatrix}$$

### Lemma 1

The equation  $\text{Det}(u) = 0$  has  $nk$  distinct solutions in  $\overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{k}}]]$

- We have

$$\text{Det}(u) = \det \begin{pmatrix} -u^{m_1} + t\epsilon^k \gamma_{1,1} u^{m_1-k} & \dots & t\epsilon^k \gamma_{1,n} u^{m_1-k} \\ \vdots & \ddots & \vdots \\ t\epsilon^k \gamma_{n,1} u^{m_n-k} & \dots & -u^{m_n} + t\epsilon^k \gamma_{n,n} u^{m_n-k} \end{pmatrix} + O(t^\alpha u^{M-nk}).$$

- Using  $\gamma_{i,i} = i^k$  and  $\gamma_{i,j} = t^\beta$ , we get  $\text{Det} = \prod_{j=1}^n (-u^k + t\epsilon^k j^k) \bmod t^{n+1}$ .
- Newton's algorithm:

$$U_i = \zeta^\ell \cdot j \cdot t^{\frac{1}{k}} + O(t^{\frac{2}{k}}), \quad \text{for } \ell = 1, \dots, k \text{ and } j = 1, \dots, n, \text{ with } \zeta^k = 1.$$

# Outline of the proof for systems in the general case

- 1 Given a system of **DDEs** for  $F_i(t, u)$  consider the perturbed system **DDE** $_{\epsilon}$ .
- 2 Prove that  $\text{Det}(u) = 0$  has  $nk$  distinct solutions  $u = U(t)$  in  $\overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{\star}}]]$ .
- 3 Define  $\mathcal{S} = (E_1, \dots, E_n, \text{Det}, P)$  and let  $\mathcal{S}_{\text{dup}}$  be the duplicated system in  $nk(n+2)$  variables.
- 4 Show:  $\langle \mathcal{S}_{\text{dup}} \rangle : \det(\text{Jac}_{\mathcal{S}_{\text{dup}}})^{\infty}$  is 0-dimensional by proving:  $\text{Jac}_{\mathcal{S}_{\text{dup}}}$  is invertible.

## Step 4: $\text{Jac}_{S_{\text{dup}}}$ is invertible (very brief sketch)

$$\text{Jac}_{S_{\text{dup}}} = \begin{pmatrix} A_1 & & 0 & B_1 \\ & \ddots & & \vdots \\ 0 & & A_{nk} & B_{nk} \end{pmatrix} \in \overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{*}}]]^{nk(n+2) \times nk(n+2)},$$

$$A_i := \begin{pmatrix} \partial_{x_1} E_1^{(i)}(U_i) & \dots & \partial_{x_n} E_1^{(i)}(U_i) & \partial_{u_i} E_1^{(i)}(U_i) \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} E_n^{(i)}(U_i) & \dots & \partial_{x_n} E_n^{(i)}(U_i) & \partial_{u_i} E_n^{(i)}(U_i) \\ \partial_{x_1} \text{Det}^{(i)}(U_i) & \dots & \partial_{x_n} \text{Det}^{(i)}(U_i) & \partial_{u_i} \text{Det}^{(i)}(U_i) \\ \partial_{x_1} P^{(i)}(U_i) & \dots & \partial_{x_n} P^{(i)}(U_i) & \partial_{u_i} P^{(i)}(U_i) \end{pmatrix}, B_i := \begin{pmatrix} \partial_{z_0} E_1^{(i)}(U_i) & \dots & \partial_{z_{nk-1}} E_1^{(i)}(U_i) \\ \vdots & \ddots & \vdots \\ \partial_{z_0} E_n^{(i)}(U_i) & \dots & \partial_{z_{nk-1}} E_n^{(i)}(U_i) \\ \partial_{z_0} \text{Det}^{(i)}(U_i) & \dots & \partial_{z_{nk-1}} \text{Det}^{(i)}(U_i) \\ \partial_{z_0} P^{(i)}(U_i) & \dots & \partial_{z_{nk-1}} P^{(i)}(U_i) \end{pmatrix}.$$

$$\Rightarrow \det(\text{Jac}_{S_{\text{dup}}}) = \pm \left( \prod_{i=1}^{nk} \det(\text{Jac}_i(U_i)) \right) \cdot \det(\Lambda), \quad \text{for}$$

$$\text{Jac}_i(u) \in \mathbb{Q}(\epsilon)[u][[t]]^{(n+1) \times (n+1)} \quad \text{and} \quad \Lambda \in \overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{*}}]]^{nk \times nk}$$

**Method:** Analyze the (lowest) valuation in  $t$  to show non-vanishing.



# Special Eulerian planar orientations

- [Bonichon, Bousquet-Mélou, Dorbec, Pennarun, 2006] consider and solve:

$$\begin{cases} F_1(t, u) = 1 + t \cdot \left( u + 2uF_1(t, u)^2 + 2uF_2(t, 1) + u \frac{F_1(t, u) - uF_1(t, 1)}{u-1} \right), \\ F_2(t, u) = t \cdot \left( 2uF_1(t, u)F_2(t, u) + uF_1(t, u) + uF_2(t, 1) + u \frac{F_2(t, u) - uF_2(t, 1)}{u-1} \right). \end{cases}$$

# Special Eulerian planar orientations

- We get polynomial equations

$$\begin{cases} E_1 = (1 - x_1) \cdot (u - 1) + t \cdot (2u^2x_1^2 - u^2z_0 + 2u^2z_1 - 2ux_1^2 + u^2 + ux_1 - 2uz_1 - u), \\ E_2 = x_2 \cdot (1 - u) + t \cdot (2u^2x_1x_2 + u^2x_1 - 2ux_1x_2 - ux_1 + ux_2 - uz_1). \end{cases}$$

- Then define

$$\begin{cases} \text{Det} = (4tu^2x_1 - 4tux_1 + tu - u + 1)(2tu^2x_1 - 2tux_1 + tu - u + 1), \\ P = -2tx_1x_2 - tx_1 + tx_2 - tz_1 - x_2 + P_1 \cdot u + P_2 \cdot u^2 + P_3 \cdot u^3, \end{cases}$$

- $\mathcal{S}_{\text{dup}} =$

$$\begin{aligned} & (E_1(x_1, x_2, z_0, z_1, u_1), E_2(x_1, x_2, z_0, z_1, u_1), \text{Det}(x_1, x_2, z_0, z_1, u_1), P(x_1, x_2, z_0, z_1, u_1), \\ & (E_1(x_3, x_4, z_0, z_1, u_2), E_2(x_3, x_4, z_0, z_1, u_2), \text{Det}(x_3, x_4, z_0, z_1, u_2), P(x_3, x_4, z_0, z_1, u_2)) \end{aligned}$$

- Compute a generator of  $\langle \mathcal{S}_{\text{dup}}, m \cdot (u_1 - u_2) - 1 \rangle \cap \mathbb{Q}[z_0, t]$ .

# Special Eulerian planar orientations in Maple

```
E1 := numer(-x1+1 + 2*t*u*x1^2 + 2*t*u*z1 + t*u*(-u*z0+x1)/(u-1)+t*u);
E2 := numer(-x2 + 2*t*u*x1*x2 + t*u*x1 + t*u*z1 + t*u*(-u*z1+x2)/(u-1));
Jac := Matrix([[diff(E1, x1),diff(E1, x2)],[diff(E2, x1),diff(E2, x2)]]);
Det := LinearAlgebra[Determinant](Jac);
Pm := Matrix([[diff(E1, x1), diff(E2, x1)], [diff(E1, u), diff(E2, u)]]);
P := LinearAlgebra[Determinant](Pm);
S := [E1, E2, det, P];
S1 := op(subs(x1=x1,x2=x2,u=u1,S));
S2 := op(subs(x1=x3,x2=x4,u=u2,S));
Sdup := [S1,S2, m*(u1 - u2) - 1];
G := polynomial_elimination(Sdup, z0, t);
```

$$(z0 - 1)(2tz0 + t - 1)(64t^3z0^3 + 48t^3z0^2 - 15t^3z0 - 72t^2z0^2 + t^3 + 9t^2z0 + \dots) \dots$$

# Summary and conclusion

- **Systems of DDEs** with one catalytic variable have algebraic solutions.
- There exists an algorithm for finding minimal polynomials of such solutions.
- Currently ongoing work on improving the efficiency and effective handling of more catalytic variables.