The art of educated algorithmic guessing (with gfun)

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Guess the sequences

- 1, 3, 9, 27, 81, 243,
- 0, 1, 3, 6, 10, 15,
- 1, 1, 2, 3, 5, 8,
- 1, 1, 2, 5, 14, 42,
- 1, 5, 73, 1445, 33001,

Guess the sequences

- 1, 3, 9, 27, 81, 243, 729 3^n
- 0, 1, 3, 6, 10, 15,
- 1, 1, 2, 3, 5, 8,
- 1, 1, 2, 5, 14, 42,
- 1, 5, 73, 1445, 33001,

Guess the sequences

- 1, 3, 9, 27, 81, 243, 729 3^n
- 0, 1, 3, 6, 10, 15, 21 $(n^2 + n)/2$
- 1, 1, 2, 3, 5, 8,
- 1, 1, 2, 5, 14, 42,
- 1, 5, 73, 1445, 33001,

Guess the sequences

- 1, 3, 9, 27, 81, 243, 729 — 3^n
- 0, 1, 3, 6, 10, 15, 21 — $(n^2 + n)/2$
- 1, 1, 2, 3, 5, 8, 13 — $F_{n+1} = F_n + F_{n-1}$
- 1, 1, 2, 5, 14, 42,
- 1, 5, 73, 1445, 33001,

Guess the sequences

- 1, 3, 9, 27, 81, 243, 729 — 3^n
- 0, 1, 3, 6, 10, 15, 21 — $(n^2+n)/2$
- 1, 1, 2, 3, 5, 8, 13 — $F_{n+1} = F_n + F_{n-1}$
- 1, 1, 2, 5, 14, 42, 132 — $\frac{1}{n+1}\binom{2n}{n}$
- 1, 5, 73, 1445, 33001,

Guess the sequences

- 1, 3, 9, 27, 81, 243, 729 — 3^n
- 0, 1, 3, 6, 10, 15, 21 — $(n^2+n)/2$
- 1, 1, 2, 3, 5, 8, 13 — $F_{n+1} = F_n + F_{n-1}$
- 1, 1, 2, 5, 14, 42, 132 — $\frac{1}{n+1}\binom{2n}{n}$
- 1, 5, 73, 1445, 33001, 819005 — $\sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$

Importance of sequences

■ Enumerative combinatorics	(e.g. counting walks in the quarter plane)
■ Algebraic number theory	(e.g. irrationality proof of $\zeta(3)$)
■ Analytic number theory	(e.g. prime number theorem)
■ Algorithmic number theory	(e.g. integer factorization)
■ Computer algebra	(e.g. fast computation of sequences)
■ Algebraic geometry	(e.g. counting points on curves)
■ Interconnections	(e.g. moonshine theory, or mirror symmetry)
■ Applications in various sciences	(e.g. uniqueness of the Clifford torus model)

P-recursive sequences and D-finite functions

- A sequence $(u_n)_{n\geq 0}$ is called P-recursive if it satisfies a linear recurrence relation with polynomial coefficients:

$$c_r(n)u_{n+r} + \cdots + c_0(n)u_n = 0 \qquad n \geq 0.$$

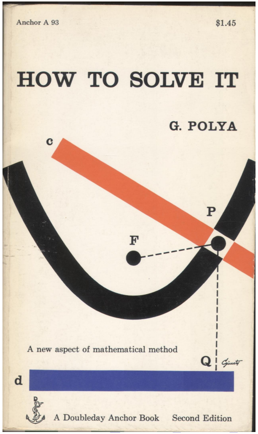
- Equivalently, the generating function $f(x) = \sum_n u_n x^n$ is D-finite, i.e. it satisfies

$$p_s(x)f^{(s)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0, \quad p_i(x) \text{ are polynomials.}$$

- Large class with very nice properties.
- There exists a vast theory for P-recursive sequences/D-finite functions.

The Guess-and-Prove approach

- Experimental mathematics and "Guess-and-Prove" propagated by G. Pólya.
- Extremely fruitful when using a computer.
- P-recursive sequences/D-finite functions: ideal data structure for guessing.
- Find new structure.
- Find simpler formulas.
- Very efficient and easy in practice with Maple.



The gfun package

- By Bruno Salvy and Paul Zimmermann in 1992. Constantly improved.
- Can guess a sequence from its first terms.
- Implemented closure properties and interplay between recursions and ODEs.
- Version 3.20 comes with Maple 2021

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with(gfun);
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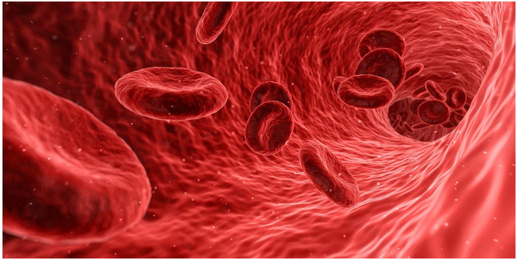
[*Laplace*, *Parameters*, *algebraicsubs*, *algeqtodiffeq*, *algeqtoseries*, *algfuntoalgeq*, *borel*, *cauchyproduct*, *'diffeq*diffeq'*, *'diffeq+diffeq'*, *diffeqtohomdiffeq*, *diffeqtorec*, *guesseqn*, *guessgf*, *hadamardproduct*, *holexprtodiffeq*, *invborel*, *listtoalgeq*, *listtodiffeq*, *listtohypergeom*, *listtolist*, *listtoratpoly*, *listtorec*, *listtoseries*, *poltodiffeq*, *poltorec*, *ratpolytocoeff*, *'rec*rec'*, *'rec+rec'*, *rectodiffeq*, *rectohomrec*, *rectoproc*, *seriestoalgeq*, *seriestodiffeq*, *seriestohypergeom*, *seriestolist*, *seriestoratpoly*, *seriestorec*, *seriestoseries*]

- Newest version (3.84): perso.ens-lyon.fr/bruno.salvy/software/the-gfun-package/.

Examples in Maple

- Toy examples.
- Monotonicity of Iso.
- Zagier's sequences.



Main takeaways

- Sequences are ubiquitous.
- Guessing allows finding structure in sequences.
- `gfun` makes efficient guessing easy.