On Rupert's problem[1]

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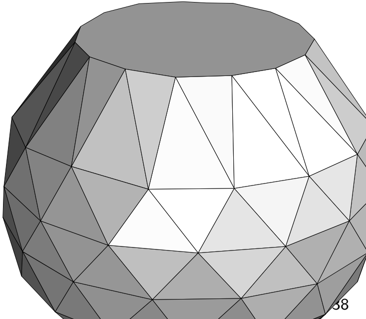
23rd October, 2025

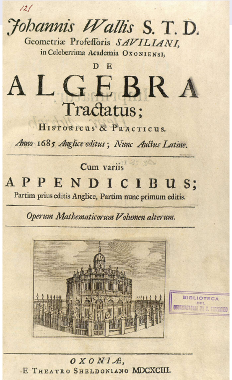


[1]Based on arxiv.org/abs/2508.18475 and arxiv.org/abs/2112.13754

Prince Rupert's cube

Fact (Wallis, 1685)

It is possible to cut a hole inside the **unit cube** such that another **unit cube** can pass through this hole.



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Johannis Wallis S. T. D.
Geometriæ Profeſſoris SAVILIANI,
in Celeberrima Academia OXONIENSI,

DE

ALGEBRA

Tractatus;

HISTORICUS & PRACTICUS.

Anno 1685 Anglice editus; Nunc Auctus Latine.

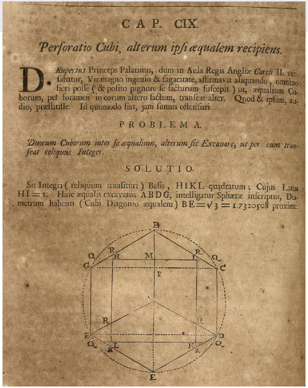
Cum variis

APPENDICIBUS;

Partim prius editis Anglice, Partim nunc primum editis.

Operum Mathematicorum Volumen alterum.

OXONIÆ,
E THEATRO SHELDONIANO MDCXCIII.



CAP. CIX.

Perforatio Cubi, alterum ipſi æqualem recipiens.

D. *Rupertus* Princeps Palatinus, dum in Aula Regis Angliæ *Caroli* II. verſabatur, Vir magno ingenio & ſagacitate, affirmavit aliquando, omnino fieri poſſe (& poſito pignore ſe facturum ſuſcepit) ut, æqualium Cuborum, per foramen in eorum altero factum, tranſeat alter. Quod & ipſum, audio, præſtitiſſe. Id quomodo fiat, jam ſumus oſtenſuri.

PROBLEMA.

Duorum Cuborum inter ſe æqualium, alterum ſic Excavare, ut per eum tranſeat reliquus Integer.

SOLUTIO.

Sit Integri (reliquum tranſituri) Baſis, HIKL quadratum; Cujus Latus HI = 1. Huic æqualis excavatus ABDG, intelligatur Sphæræ inſcriptus, Diametrum habenti (Cubi Diagonio æqualem) BE = $\sqrt{3}$ = 1.7320508 proxime.

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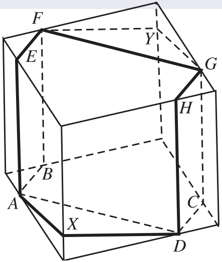
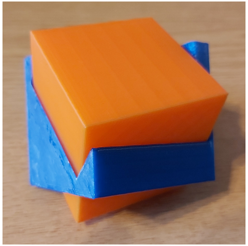




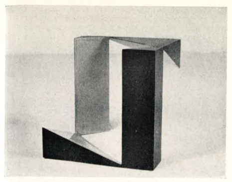


Fig. 4a

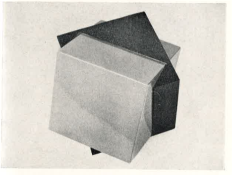
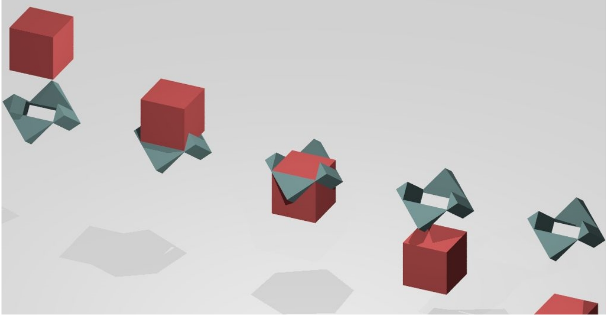


Fig. 4b

- How to understand this paradox? How to find the solution?
- Which other solids have this property? Is there a characterization?
- What is the "optimal" way to put a cube inside a cube? What about other solids?

Rupert's problem

- A convex polyhedron **P** is called *Rupert* if a hole with the shape of a straight tunnel can be cut into it such that a copy of **P** can be moved through this hole.
- *Rupert's problem* is the task to decide whether a given polyhedron is Rupert.



Nieuwland's number

Fact (Nieuwland, 1816)

It is possible to cut a hole inside the **unit cube** such that a **cube with side length less than** $3\sqrt{2}/4 \approx 1.0606$ can pass through this hole.

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- The largest number $\nu \in \mathbb{R}$ such that $\nu\mathbf{P}$ passes through some hole inside $\mathbf{P}$ is called *Nieuwland number of* $\mathbf{P}$.
- For all solids it holds that $\nu \geq 1$.
- $\mathbf{P}$ is Rupert $\iff \nu(\mathbf{P}) > 1$.
- ν of the Cube is exactly $3\sqrt{2}/4$ [Nieuwland, 1816].

Brief history of Rupert's problem

- The Cube is Rupert [conjectured by Prince Rupert, proved by Wallis 1685].
- The Nieuwland number of the Cube is $3\sqrt{2}/4$ [Nieuwland, 1816].

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- 9 of 13 Archimedean solids are Rupert [Hoffmann, 2018] [Lavau, 2019].

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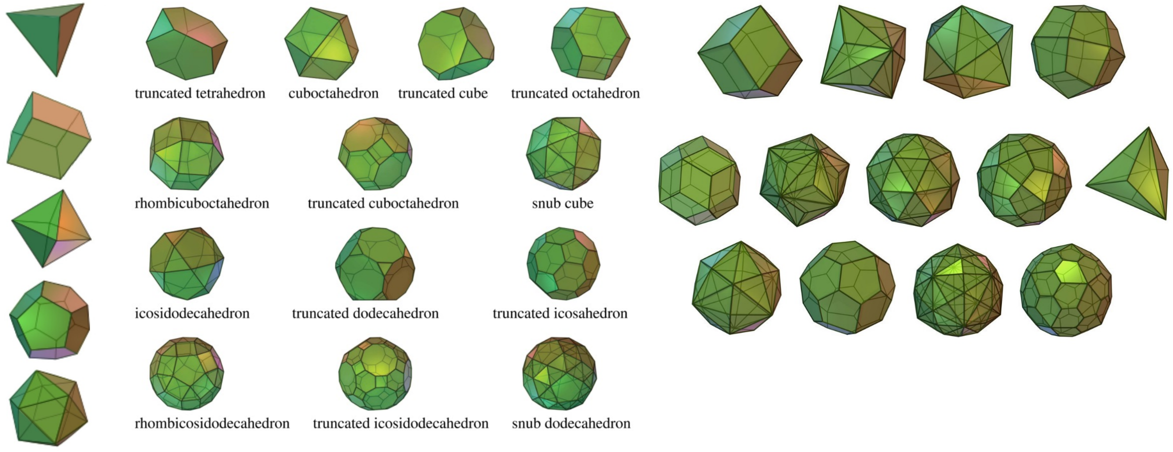
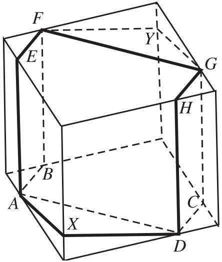
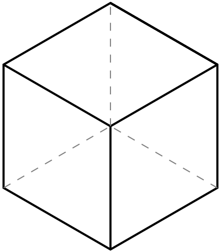
- 9 of 13 Archimedean solids are Rupert [Hoffmann, 2018] [Lavau, 2019].
- 10 of 13 Archimedean solids and many other polyhedra are Rupert. Efficient way for proving Rupert's property. Theoretical algorithm for deciding. [S., Y., 2021].
- 11 of 13 Catalan solids are Rupert, improved optimization [Fredriksson, 2022].
- The **Noperthedron**: a counter example to the conjecture [S., Y., 2025].

Solids


truncated tetrahedron
cuboctahedron
truncated cube
truncated octahedron
rhombicuboctahedron
truncated cuboctahedron
snub cube
icosidodecahedron
truncated dodecahedron
truncated icosahedron
rhombicosidodecahedron
truncated icosidodecahedron
snub dodecahedron

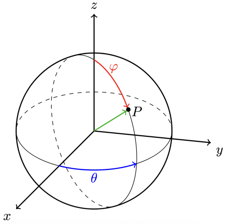
Definition of Rupert's problem

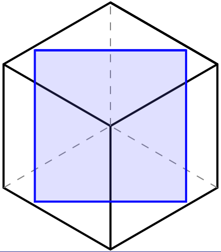




Definition of Rupert's problem

Let $M(\theta, \varphi) : \mathbb{R}^3 \to \mathbb{R}^2$ be an orthogonal projection map in direction $X(\theta, \varphi) \in \mathbb{R}^3$ and $R(\alpha) : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation map.





$$X(\theta, \varphi) := (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)^t,$$

$$M(\theta, \varphi) := \begin{pmatrix} -\sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta)\cos(\varphi) & -\sin(\theta)\cos(\varphi) & \sin(\varphi), \end{pmatrix},$$

$$R(\alpha) := \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Definition

A point-symmetric polyhedron **P** has Rupert's property, if there exist 5 parameters $\alpha, \theta_1, \theta_2 \in [0, 2\pi)$ and $\varphi_1, \varphi_2 \in [0, \pi]$ such that

$$R(\alpha) \circ M(\theta_1, \varphi_1)\mathbf{P} \subset (M(\theta_2, \varphi_2)\mathbf{P})^\circ.$$

A basic (Las Vegas) algorithm

Input: A centrally symmetric polyhedron **P**.
Output: The solution encoded by $\Psi = (\alpha, \theta_1, \theta_2, \varphi_1, \varphi_2) \in \mathbb{R}^5$ if **P** is Rupert.

(1) Draw θ_1, θ_2 and α uniformly in $[0, 2\pi)$, and φ_1, φ_2 uniformly in $[0, \pi]$.

(2) Construct the two 3×2 matrices A and B corresponding to the linear maps $R(\alpha) \circ M(\theta_1, \varphi_1)$ and $M(\theta_2, \varphi_2)$. Compute the two projections of **P** given by $\mathcal{P}' := A \cdot \mathbf{P}$ and $\mathcal{Q}' := B \cdot \mathbf{P}$.

(3) Find vertices on the convex hulls of $\mathcal{P}'$ and $\mathcal{Q}'$; denote them by $\mathcal{P}$ and $\mathcal{Q}$.

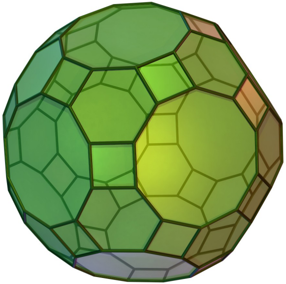
(4) Decide whether $\mathcal{P}$ lies inside of $\mathcal{Q}$ by checking each vertex of $\mathcal{P}$.

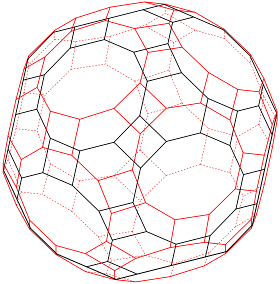
(5) If Step (4) yields a True, return the solution $\Psi = (\alpha, \theta_1, \theta_2, \varphi_1, \varphi_2)$. Otherwise, repeat Steps (1)-(5).

The Truncated Icosidodecahedron

Theorem (S., Y., 2021)

The Truncated Icosidodecahedron has Rupert's property.





The Truncated Icosidodecahedron

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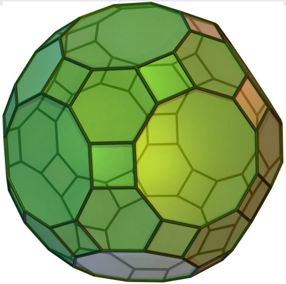
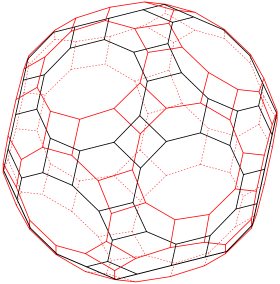
Proof.

$\alpha = 0.43584, \theta_1 = 2.77685, \theta_2 = 0.79061, \varphi_1 = 2.09416, \varphi_2 = 2.89674,$
plus some verification of linear inequalities in Maple/SageMath. $\square$



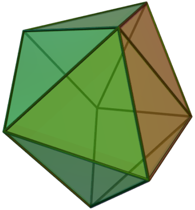
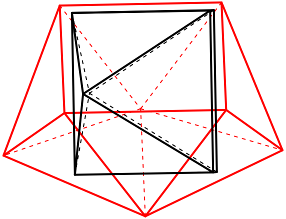
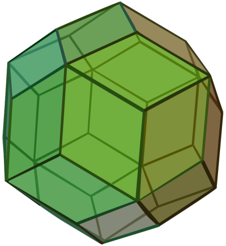
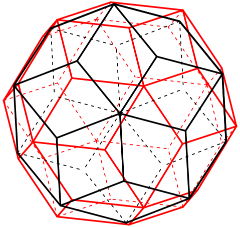


Platonic, Archimedean, Catalan and Johnson Solids

Theorem (S., Y., 2021)

In a few minutes it can be proven automatically that:

1. *All 5 Platonic solids are Rupert.*
2. *At least 10 out of 13 Archimedean solids are Rupert.*
3. *At least 9 out of 13 Catalan solids are Rupert.*
4. *At least 82 out of 92 Johnson solids are Rupert.*

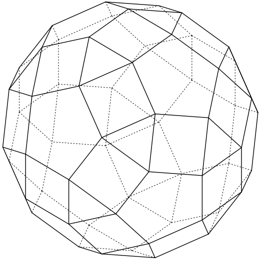





Are all convex polyhedra Rupert?

Conjecture (Jerrard, Wetzel, Yuan, 2017 and Chai, Yuan, Zamfirescu, 2018)

All convex polyhedra have Rupert's property.

- All Platonic solids are Rupert.
- 3 Archimedean solids remain open. One of them is point-symmetric.



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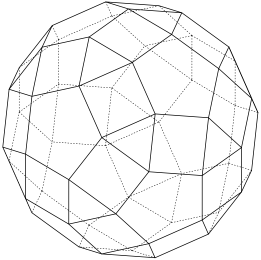

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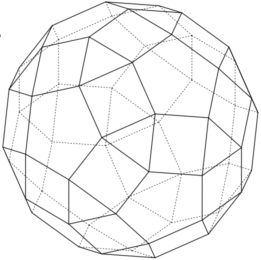
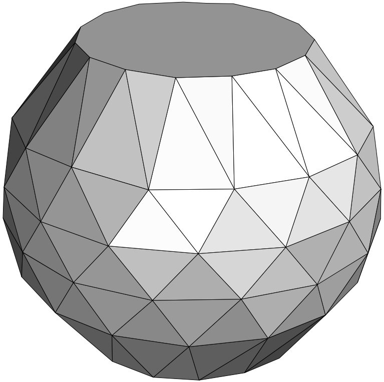
- 3 Archimedean solids remain open. One of them is point-symmetric.
- Rhombicosidodecahedron natural candidate for disproving conjecture.
- *Rupertness*: Probability that a random projection yields a solution.
- Can estimate confidence intervals for Rupertness.
- Conclusion: RID is significantly different from other regular solids. It is likely that RID is a counter example to the conjecture.



4 years later: The Noperthedron is not Rupert

Theorem (S., Y., 2025)

The Noperthedron, **NOP**, *does not have Rupert's property.*



Idea of proof

- Partition the five-dimensional solution space

$$I = [0, 2\pi) \times [0, \pi] \times [0, 2\pi) \times [0, \pi] \times [-\pi, \pi)$$

  into small parts and prove for each that no solution in that region exists.
- Rough idea: Show that the "middle point" of any region does not yield a solution and argue with effective continuity of the parameters that this also excludes an explicit neighborhood around that point.

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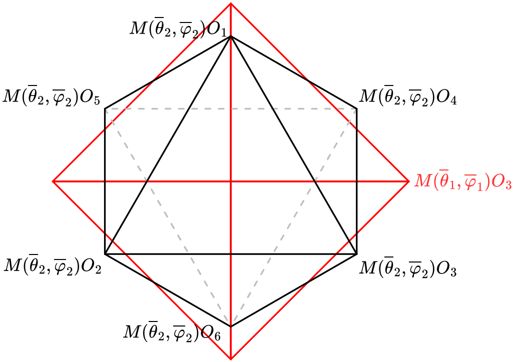
  into small parts and prove for each that no solution in that region exists.
- Rough idea: Show that the "middle point" of any region does not yield a solution and argue with effective continuity of the parameters that this also excludes an explicit neighborhood around that point.
- The **global theorem** is tailored for two generic projections of $\mathbf{P}$, when some vertex of the "smaller" projection $\mathcal{P} = R(\alpha)M(\theta_1, \varphi_1)\,\mathbf{P}$ is *strictly outside* the "larger" projection $\mathcal{Q} = M(\theta_2, \varphi_2)\,\mathbf{P}$.
- The **local theorem** can handle projections that look (almost) exactly the same, for instance if $\theta_1 \approx \theta_2$, $\varphi_1 \approx \varphi_2$ and $\alpha \approx 0$.

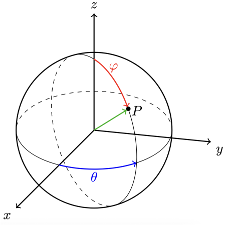
Motivation for global theorem



$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) = (0, 0, \pi/4, \tan^{-1}(\sqrt{2}), 0)$$

Starting point of the global theorem

Recall:



$$X(\theta,\varphi) := (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)^t,$$

$$M(\theta,\varphi) := \begin{pmatrix} -\sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta)\cos(\varphi) & -\sin(\theta)\cos(\varphi) & \sin(\varphi), \end{pmatrix},$$

$$R(\alpha) := \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Lemma

Let $\varepsilon > 0$ *and* $|\theta - \overline{\theta}|, |\varphi - \overline{\varphi}|, |\alpha - \overline{\alpha}| \leq \varepsilon$ *then*

- $\|M(\theta,\varphi) - M(\overline{\theta},\overline{\varphi})\| < \sqrt{2}\varepsilon,$
- $\|X(\theta,\varphi) - X(\overline{\theta},\overline{\varphi})\| < \sqrt{2}\varepsilon,$
- $\|R(\alpha)M(\theta,\varphi) - R(\overline{\alpha})M(\overline{\theta},\overline{\varphi})\| < \sqrt{5}\varepsilon.$

Proof

$$R_x(\alpha) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, R_y(\alpha) := \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}, R_z(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X(\theta,\varphi)^t = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \cdot R_y(\varphi) \cdot R_z(-\theta), \quad M(\theta,\varphi) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \cdot R_y(\varphi) \cdot R_z(-\theta)$$

$$R(\alpha)M(\theta,\varphi) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \cdot R_z(\alpha) \cdot R_y(\varphi) \cdot R_z(-\theta).$$

Lemma

For any $\alpha, \beta \in \mathbb{R}$ one has $\|R_x(\alpha)R_y(\beta) - \mathrm{Id}\| \leq \sqrt{\alpha^2 + \beta^2}$.

Sketch of proof: Write $R_x(\alpha)R_y(\beta) = UR_x(\Phi)U^{-1}$ and take trace to obtain:

$$\cos(\alpha) + \cos(\beta) + \cos(\alpha)\cos(\beta) = 1 + 2\cos(\Phi).$$

Jensen on $f(t) = \cos(\sqrt{t})$ shows LHS $\geq 1 + 2\cos(\sqrt{\alpha^2 + \beta^2})$, thus $|\Phi| \leq \sqrt{\alpha^2 + \beta^2}$.

First version of the global theorem

Recall the definition of Rupert's property

$$R(\alpha)M(\theta_1,\varphi_1)\,\mathbf{P} \subset (M(\theta_2,\varphi_2)\,\mathbf{P})^\circ.$$

If $\Psi \in \mathbb{R}^5$ is a solution, then for any vector $w \in \mathbb{R}^2$ it holds that

$$\langle R(\alpha)M(\theta_1,\varphi_1)S, w\rangle < \max_{P\in\mathbf{P}}\langle M(\theta_2,\varphi_2)P, w\rangle.$$

Theorem (Global theorem v0.1)

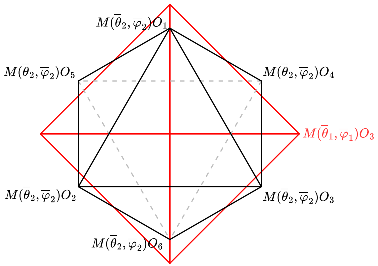
Let $\mathbf{P}$ *be convex, pointsymmetric with radius* 1*. Assume:*

$$\langle R(\overline{\alpha})M(\overline{\theta}_1,\overline{\varphi}_1)S, w\rangle > \max_{P\in\mathbf{P}}\langle M(\overline{\theta}_2,\overline{\varphi}_2)P, w\rangle + (\sqrt{2}+\sqrt{5})\varepsilon$$

for some $S \in \mathbf{P}$ *and* $w \in \mathbb{R}^2$ *with* $\|w\| = 1$*, then there cannot be a solution* $(\theta_1,\varphi_1,\theta_2,\varphi_2,\alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon]$.

Example

- Consider the Octahedron $\mathbf{O} = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\} \subseteq \mathbb{R}^3$ and two projection directions $(\overline{\theta}_1, \overline{\varphi}_1) = (0, 0)$ and $(\overline{\theta}_2, \overline{\varphi}_2) = (\pi/4, \tan^{-1}(\sqrt{2}))$. Set $\overline{\alpha} = 0$.
- Goal: show this is no solution to Rupert's problem and there exists $\varepsilon > 0$ such that there is also no solution $(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha)$ with $|\overline{\theta}_i - \theta_i|, |\overline{\varphi}_i - \varphi_i|, |\alpha| \leq \varepsilon$.



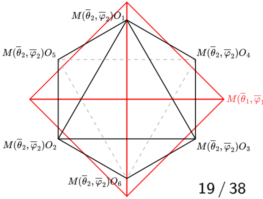
Example continued

Following the global theorem we first compute:

$$M(\overline{\theta}_1, \overline{\varphi}_1) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M(\overline{\theta}_2, \overline{\varphi}_2) = \begin{pmatrix} -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{6}/6 & -\sqrt{6}/6 & \sqrt{6}/3 \end{pmatrix}.$$

We choose $S = O_3 = (0,1,0)$ and $w = (1,0)$, thus $\langle R(\overline{\alpha}) M(\overline{\theta}_1, \overline{\varphi}_1) S, w \rangle = 1$.

$$\langle M(\overline{\theta}_2, \overline{\varphi}_2) P, w \rangle = \begin{cases} 0 & \text{for } P = M(\overline{\theta}_2, \overline{\varphi}_2) O_i \;\; \text{with} \;\; i = 1, 6, \\ \sqrt{2}/2 & \text{for } P = M(\overline{\theta}_2, \overline{\varphi}_2) O_i \;\; \text{with} \;\; i = 3, 4, \\ -\sqrt{2}/2 & \text{for } P = M(\overline{\theta}_2, \overline{\varphi}_2) O_i \;\; \text{with} \;\; i = 2, 5. \end{cases}$$



Example continued

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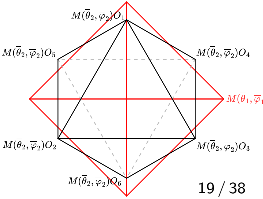
Thus if $\varepsilon > 0$ is chosen so that

$$1 > \sqrt{2}/2 + (\sqrt{2} + \sqrt{5})\varepsilon, \qquad \text{e.g., } \varepsilon = 0.08$$

there is no solution in

$$(0, 0, \pi/4, \tan^{-1}(\sqrt{2}), 0) \pm \varepsilon \subseteq \mathbb{R}^5.$$



Actual Global theorem

Theorem (Global Theorem v1.0)

Let $\mathbf{P}$ *be a pointsymmetric convex polyhedron with radius* 1 *and let* $S \in \mathbf{P}$. *Let* $w \in \mathbb{R}^2$ *be a unit vector and denote* $\overline{M_1} := M(\overline{\theta}_1, \overline{\varphi}_1)$, $\overline{M_2} := M(\overline{\theta}_2, \overline{\varphi}_2)$ *as well as* $\overline{M_1}^{\theta} := M^{\theta}(\overline{\theta}_1, \overline{\varphi}_1)$, $\overline{M_1}^{\varphi} := M^{\varphi}(\overline{\theta}_1, \overline{\varphi}_1)$ *and analogously for* $\overline{M_2}^{\theta}, \overline{M_2}^{\varphi}$. *Finally set*

$$G := \langle R(\overline{\alpha})\overline{M_1}S, w\rangle - \varepsilon \cdot \left(|\langle R'(\overline{\alpha})\overline{M_1}S, w\rangle| + |\langle R(\overline{\alpha})\overline{M_1}^{\theta}S, w\rangle| + |\langle R(\overline{\alpha})\overline{M_1}^{\varphi}S, w\rangle|\right) - 9\varepsilon^2/2,$$

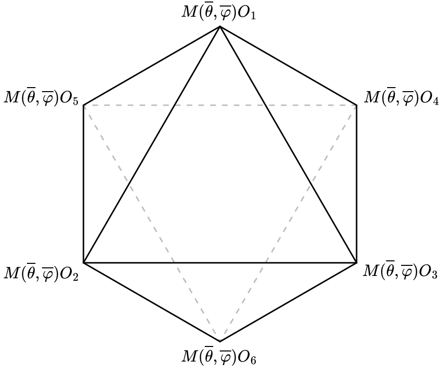
$$H_P := \langle \overline{M_2}P, w\rangle + \varepsilon \cdot \left(|\langle \overline{M_2}^{\theta}P, w\rangle| + |\langle \overline{M_2}^{\varphi}P, w\rangle|\right) + 2\varepsilon^2, \qquad \text{for } P \in \mathbf{P}.$$

If $G > \max_{P \in \mathbf{P}} H_P$ *then there does not exist a solution to Rupert's condition with*

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon] \subseteq \mathbb{R}^5.$$

$\varepsilon = 0.08$ from Example can be replaced by $\varepsilon = 0.164$

Motivation for local theorem: $\theta_1 = \theta_2$, $\varphi_1 = \varphi_2$, $\alpha = 0$



$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) = (\pi/4, \tan^{-1}(\sqrt{2}), \pi/4, \tan^{-1}(\sqrt{2}), 0)$$

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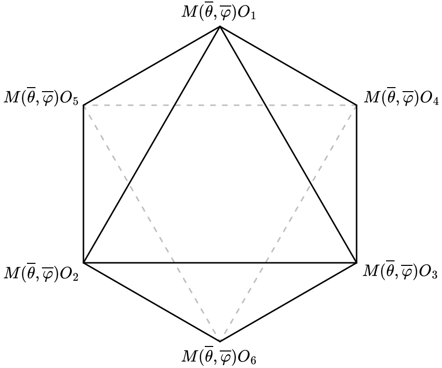

$$A := M(\theta, \varphi)O_1,$$
$$B := M(\theta, \varphi)O_2,$$
$$C := M(\theta, \varphi)O_3$$

Fact:

$$\|A\|^2 + \|B\|^2 + \|C\|^2 = 2$$

$\Longrightarrow$ no local solution from this direction

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) = (\pi/4, \tan^{-1}(\sqrt{2}), \pi/4, \tan^{-1}(\sqrt{2}), 0)$$

Conditions for the three-point method

- P_1, P_2, P_3 are all in front of the projection:

$$\langle X(\overline{\theta}, \overline{\varphi}), P_i \rangle > 0.$$

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$$\| M(\bar{\theta}, \overline{\varphi}) P_i \| > r.$$

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- The points P_1, P_2, P_3 are locally maximally distant.

Conditions for the three-point method

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$$\langle X(\bar{\theta}, \overline{\varphi}), P_i \rangle > \sqrt{2}\varepsilon.$$

- The origin is inside the projected triangle $M(\bar{\theta}, \overline{\varphi}) \cdot \{P_1, P_2, P_3\}$:

$$\langle R(\pi/2)M(\bar{\theta}, \overline{\varphi})P_1, M(\bar{\theta}, \overline{\varphi})P_2 \rangle > \sqrt{2}\varepsilon(\sqrt{2} + \varepsilon),$$
$$\langle R(\pi/2)M(\bar{\theta}, \overline{\varphi})P_2, M(\bar{\theta}, \overline{\varphi})P_3 \rangle > \sqrt{2}\varepsilon(\sqrt{2} + \varepsilon),$$
$$\langle R(\pi/2)M(\bar{\theta}, \overline{\varphi})P_3, M(\bar{\theta}, \overline{\varphi})P_1 \rangle > \sqrt{2}\varepsilon(\sqrt{2} + \varepsilon).$$

- The projected points $M(\bar{\theta}, \overline{\varphi}) \cdot \{P_1, P_2, P_3\}$ are not to close to the origin:

$$\| M(\bar{\theta}, \overline{\varphi}) P_i \| > r + \sqrt{2}\varepsilon.$$

- The points P_1, P_2, P_3 are $\sqrt{5}\varepsilon$-locally maximally distant.

ε-spanning points

Definition

Three points P_1, P_2, P_3 with $\|P_1\|, \|P_2\|, \|P_3\| \leq 1$ are called *ε-spanning for* (θ, φ) if:

$$\langle R(\pi/2)M(\theta,\varphi)P_1, M(\theta,\varphi)P_2\rangle > \sqrt{2}\varepsilon(\sqrt{2}+\varepsilon).$$
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$$\langle R(\pi/2)M(\theta,\varphi)P_3, M(\theta,\varphi)P_1\rangle > \sqrt{2}\varepsilon(\sqrt{2}+\varepsilon).$$

Lemma

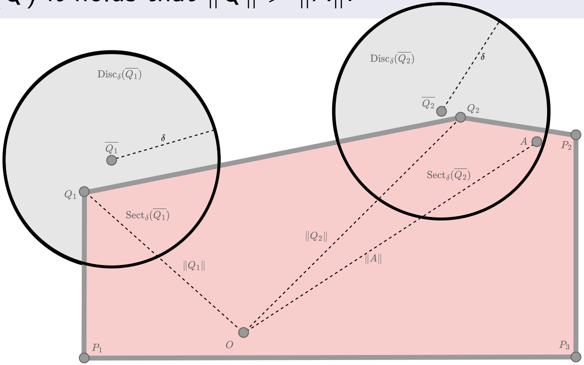
Let P_1, P_2, P_3 with $\|P_1\|, \|P_2\|, \|P_3\| \leq 1$ be ε-spanning for $(\overline{\theta}, \overline{\varphi})$ and let $\theta, \varphi \in \mathbb{R}$ such that $|\theta - \overline{\theta}|, |\varphi - \overline{\varphi}| \leq \varepsilon$. Assume that $\langle X(\theta,\varphi), P_i\rangle > 0$ for $i = 1, 2, 3$. Then

$$X(\theta,\varphi) \in \operatorname{span}^+(P_1, P_2, P_3).$$

Locally maximally distant points

Definition

Let $\mathcal{P} \subset \mathbb{R}^2$ be a convex polygon and $Q \in \mathcal{P}$. Assume that for some $\overline{Q} \in \mathbb{R}^2$ it holds that $Q \in \mathrm{Disc}_\delta(\overline{Q})$, i.e. $\|Q - \overline{Q}\| < \delta$. Define $\mathrm{Sect}_\delta(\overline{Q}) := \mathrm{Disc}_\delta(\overline{Q}) \cap \mathcal{P}^\circ$. Moreover, $Q \in \mathcal{P}$ is called *δ-locally maximally distant with respect to $\overline{Q}$ (δ-LMD($\overline{Q}$))* if for all $A \in \mathrm{Sect}_\delta(\overline{Q})$ it holds that $\|Q\| > \|A\|$.



Sufficient condition of LMD

Lemma

Let $\mathcal{P}$ be a convex polygon and $Q \in \mathcal{P}$. Let $\overline{Q} \in \mathbb{R}^2$ with $\|Q - \overline{Q}\| < \delta$ for some $\delta > 0$. Assume that for some $r > 0$ such that $\|Q\| > r$ it holds that

$$\frac{\langle Q, Q - P_j \rangle}{\|Q\| \|Q - P_j\|} \geq \frac{\delta}{r},$$

for all vertices $P_j \in \mathcal{P} \setminus Q$. Then $Q \in \mathcal{P}$ is δ-LMD$(\overline{Q})$.

Sufficient condition of LMD

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Let $\mathcal{P}$ be a convex polygon and $Q \in \mathcal{P}$. Let $\overline{Q} \in \mathbb{R}^2$ with $\|Q - \overline{Q}\| < \delta$ for some $\delta > 0$. Assume that for some $r > 0$ such that $\|Q\| > r$ it holds that

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for all vertices $P_j \in \mathcal{P} \setminus Q$. Then $Q \in \mathcal{P}$ is δ-LMD$(\overline{Q})$.

Proof sketch.

Assume $A \in \mathrm{Sect}_\delta(\overline{Q}) = \mathrm{Disc}_\delta(\overline{Q}) \cap \mathcal{P}^\circ$, use $\cos(\angle(O, Q, P_j)) = \frac{\langle Q, Q - P_j \rangle}{\|Q\| \|Q - P_j\|} \geq \frac{\delta}{r}$ to conclude that $\cos(\angle(O, Q, A)) \geq \delta / r$. Therefore,

$$\|A\|^2 - \|Q\|^2 = \|Q - A\| \cdot (\|Q - A\| - 2\|Q\| \cos(\angle(O, Q, A))) < 0. \qquad \square$$

Local Theorem

Theorem (Local Theorem v0.1)

Let $\mathbf{P}$ *be a convex, pointsymmetric polyhedron with radius 1 and* $P_1, P_2, P_3 \in \mathbf{P}$. *Set* $\overline{X} := X(\overline{\theta}, \overline{\varphi})$, $\overline{M} := M(\overline{\theta}, \overline{\varphi})$. *Assume that*

$$\langle \overline{X}, P_i \rangle > \sqrt{2}\varepsilon \tag{A_ε}$$

for all $i = 1, 2, 3$. *Moreover, assume that* P_1, P_2, P_3 *are* ε*-spanning for* $(\overline{\theta}, \overline{\varphi})$. *Finally, assume that for all* $i = 1, 2, 3$ *and any* $P_j \in \mathbf{P} \setminus P_i$ *it holds that*

$$\frac{\langle \overline{M} P_i, \overline{M}(P_i - P_j) \rangle - 2\varepsilon \| P_i - P_j \| \cdot (\sqrt{2} + \varepsilon)}{(\| \overline{M} P_i \| + \sqrt{2}\varepsilon) \cdot (\| \overline{M}(P_i - P_j) \| + 2\sqrt{2}\varepsilon)} > \frac{\sqrt{5}\varepsilon}{r}, \tag{B_ε}$$

for some $r > 0$ *such that* $\min_{i=1,2,3} \| \overline{M} P_i \| > r + \sqrt{2}\varepsilon$. *Then there exists no solution to Rupert's problem with* $(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon, \overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon, \pm\varepsilon]$.

Some more lemmas

Lemma (Pythagoras)

For any $P \in \mathbb{R}^3$ *one has* $\left\|M(\theta,\varphi)P\right\|^2 = \|P\|^2 - \langle X(\theta,\varphi), P\rangle^2$.

Lemma (Trinity)

Let $V_1, V_2, V_3, Y, Z \in \mathbb{R}^3$ *with* $\|Y\| = \|Z\|$ *and* $Y, Z \in \operatorname{span}^+(V_1, V_2, V_3)$*. Then at least one of the following inequalities does not hold:*

$$\begin{aligned}\langle V_1, Y\rangle &> \langle V_1, Z\rangle,\\ \langle V_2, Y\rangle &> \langle V_2, Z\rangle,\\ \langle V_3, Y\rangle &> \langle V_3, Z\rangle.\end{aligned}$$

Proof sketch of the Local Theorem

Proof sketch.

1. Assume $(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon, \overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon, \pm\varepsilon]$, let

$$M_1 = M(\theta_1, \varphi_1), M_2 = M(\theta_2, \varphi_2), X_1 = X(\theta_1, \varphi_1), X_2 = X(\theta_2, \varphi_2).$$

Proof sketch of the Local Theorem

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2. $(\mathrm{B}_\varepsilon) \Rightarrow M_2 P_i$ is $\sqrt{5}\varepsilon$-LMD wrt. $\overline{M} P_i$.

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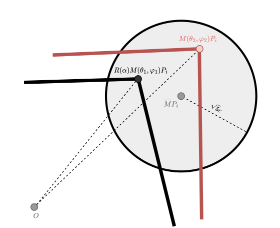
2. $(\mathrm{B}_\varepsilon) \Rightarrow M_2 P_i$ is $\sqrt{5}\varepsilon$-LMD wrt. $\overline{M} P_i$.
3. Rupert's property $+ \sqrt{5}\varepsilon$-LMD $\Rightarrow \|M_2 P_i\| > \|M_1 P_i\|$ for $i = 1, 2, 3$.



Proof sketch of the Local Theorem

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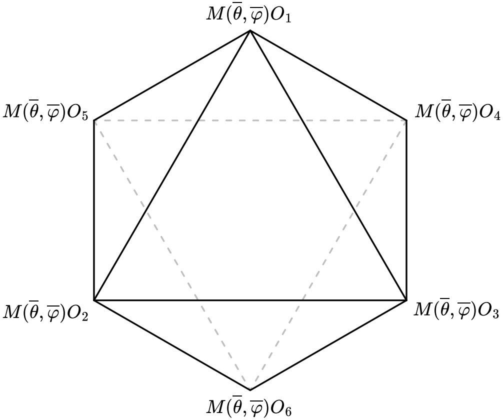
3. Rupert's property + $\sqrt{5}\varepsilon$-LMD $\Rightarrow \|M_2 P_i\| > \|M_1 P_i\|$ for $i = 1, 2, 3$.
4. $(\mathrm{A}_\varepsilon) \Rightarrow \langle X_1, P_i \rangle, \langle X_2, P_i \rangle > 0$.
5. (3), (4) and (Pythagoras) $\Rightarrow \langle X_1, P_i \rangle > \langle X_2, P_i \rangle$
6. P_1, P_2, P_3 are ε-spanning (+ Lemma)$\Rightarrow X_1, X_2 \in \mathrm{span}^+(P_1, P_2, P_3)$
7. (5)+(6) + Trinity lemma $\Rightarrow$ contradiction.

□

Example continued



Theorem (Local Theorem v1.0)

Let $\mathbf{P}$ *be a polyhedron with radius* 1 *and* $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in \mathbf{P}$*. Assume that* P_1, P_2, P_3 *and* Q_1, Q_2, Q_3 *are congruent. Let* $\varepsilon > 0$ *and* $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{R}$*, then set* $\overline{X_1} := X(\overline{\theta}_1, \overline{\varphi}_1), \overline{X_2} := X(\overline{\theta}_2, \overline{\varphi}_2)$ *as well as* $\overline{M_1} := M(\overline{\theta}_1, \overline{\varphi}_1), \overline{M_2} := M(\overline{\theta}_2, \overline{\varphi}_2)$*. Assume that there exist* $\sigma_P, \sigma_Q \in \{0, 1\}$ *such that*

$$(-1)^{\sigma_P} \langle \overline{X_1}, P_i \rangle > \sqrt{2}\varepsilon \quad \text{and} \quad (-1)^{\sigma_Q} \langle \overline{X_2}, Q_i \rangle > \sqrt{2}\varepsilon, \tag{A_ε}$$

for all $i = 1, 2, 3$*. Moreover, assume that* P_1, P_2, P_3 *are* ε*-spanning for* $(\overline{\theta}_1, \overline{\varphi}_1)$ *and that* Q_1, Q_2, Q_3 *are* ε*-spanning for* $(\overline{\theta}_2, \overline{\varphi}_2)$*. Finally, assume that for all* $i = 1, 2, 3$ *and any* $Q_j \in \mathbf{P} \setminus Q_i$ *it holds that*

$$\frac{\langle \overline{M_2} Q_i, \overline{M_2}(Q_i - Q_j) \rangle - 2\varepsilon \|Q_i - Q_j\| \cdot (\sqrt{2} + \varepsilon)}{(\|\overline{M_2} Q_i\| + \sqrt{2}\varepsilon) \cdot (\|\overline{M_2}(Q_i - Q_j)\| + 2\sqrt{2}\varepsilon)} > \frac{\sqrt{5}\varepsilon + \delta}{r}, \tag{B_ε}$$

for some $r > 0$ *such that* $\min_{i=1,2,3} \|\overline{M_2} Q_i\| > r + \sqrt{2}\varepsilon$ *and for some* $\delta \in \mathbb{R}$ *with*

$$\delta \geq \max_{i=1,2,3} \left\| R(\overline{\alpha}) \overline{M_1} P_i - \overline{M_2} Q_i \right\| / 2.$$

Then there exists no solution to Rupert's problem $R(\alpha) M(\theta_1, \varphi_1) \mathbf{P} \subset M(\theta_2, \varphi_2) \mathbf{P}^\circ$ *with*

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon] \subseteq \mathbb{R}^5.$$

Motivation for rational approximation

```
R 4.5.1 · ~/
> 1 + 10^(-16) > 1
[1] FALSE
>
```

```
SageMath version 10.0, Release Date: 2023-05-20
Using Python 3.11.1. Type "help()" for help.

sage: 1 + 10^(-16) > 1
True
sage:
```

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sage: 1 + 10^(-16) > 1
True
sage:
```

imprecise but fast

exact but slow

Idea of rational approximation $\kappa := 10^{-10}$

$$\sin_{\mathbb{Q}}(x) := x - \frac{x^3}{3} \mp + \frac{x^5}{5!} \cdots + \frac{x^{25}}{25!},$$
$$\cos_{\mathbb{Q}}(x) := 1 - \frac{x^2}{2} \mp + \frac{x^4}{4!} \cdots + \frac{x^{24}}{24!}.$$

By replacing $\sin, \cos$ with $\sin_{\mathbb{Q}}, \cos_{\mathbb{Q}}$ define the functions

$$R_{\mathbb{Q}}(\alpha), R'_{\mathbb{Q}}(\alpha), X_{\mathbb{Q}}(\theta,\varphi), M_{\mathbb{Q}}(\theta,\varphi), M^{\theta}_{\mathbb{Q}}(\theta,\varphi), M^{\varphi}_{\mathbb{Q}}(\theta,\varphi).$$

Idea of rational approximation $\kappa := 10^{-10}$

$$\sin_{\mathbb{Q}}(x) := x - \frac{x^3}{3} \mp + \frac{x^5}{5!} \cdots + \frac{x^{25}}{25!},$$
$$\cos_{\mathbb{Q}}(x) := 1 - \frac{x^2}{2} \mp + \frac{x^4}{4!} \cdots + \frac{x^{24}}{24!}.$$

By replacing $\sin, \cos$ with $\sin_{\mathbb{Q}}, \cos_{\mathbb{Q}}$ define the functions

$$R_{\mathbb{Q}}(\alpha), R'_{\mathbb{Q}}(\alpha), X_{\mathbb{Q}}(\theta, \varphi), M_{\mathbb{Q}}(\theta, \varphi), M^{\theta}_{\mathbb{Q}}(\theta, \varphi), M^{\varphi}_{\mathbb{Q}}(\theta, \varphi).$$

Lemma

Let $\alpha, \theta, \varphi \in [-4, 4]$. Then it holds that

$$\|R(\alpha) - R_{\mathbb{Q}}(\alpha)\|, \|R'(\alpha) - R'_{\mathbb{Q}}(\alpha)\|, \|X(\theta, \varphi) - X_{\mathbb{Q}}(\theta, \varphi)\|, \|M(\theta, \varphi) - M_{\mathbb{Q}}(\theta, \varphi)\| \leq \kappa.$$

Moreover,

$$\|R_{\mathbb{Q}}(\alpha)\|, \|R'_{\mathbb{Q}}(\alpha)\|, \|X_{\mathbb{Q}}(\theta, \varphi)\|, \|M_{\mathbb{Q}}(\theta, \varphi)\| \leq 1 + \kappa.$$

Rational global theorem

Theorem (Rational Global Theorem)

Let $\mathbf{P}$ *be a pointsymmetric convex polyhedron with radius* $\rho = 1$ *and* $\widetilde{\mathbf{P}}$ *a* κ*-rational approximation. Let* $\widetilde{S} \in \widetilde{\mathbf{P}}$*. Further let* $\varepsilon > 0$ *and* $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{Q} \cap [-4, 4]$*. Let* $w \in \mathbb{Q}^2$ *be a unit vector. Denote* $\overline{M_1} := M_{\mathbb{Q}}(\overline{\theta}_1, \overline{\varphi}_1)$*,* $\overline{M_2} := M_{\mathbb{Q}}(\overline{\theta}_2, \overline{\varphi}_2)$ *as well as* $\overline{M_1}^\theta := M_{\mathbb{Q}}^\theta(\overline{\theta}_1, \overline{\varphi}_1)$*,* $\overline{M_1}^\varphi := M_{\mathbb{Q}}^\varphi(\overline{\theta}_1, \overline{\varphi}_1)$ *and analogously for* $\overline{M_2}^\theta, \overline{M_2}^\varphi$*. Finally set*

$$G^{\mathbb{Q}} := \langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}\widetilde{S}, w\rangle - \varepsilon \cdot \left(|\langle R'_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}\widetilde{S}, w\rangle| + |\langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}^\theta\widetilde{S}, w\rangle| + |\langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}^\varphi\widetilde{S}, w\rangle|\right) - 9\varepsilon^2/2 - 4\kappa(1+3\varepsilon),$$

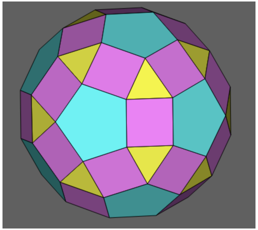
$$H_P^{\mathbb{Q}} := \langle \overline{M_2}P, w\rangle + \varepsilon \cdot \left(|\langle \overline{M_2}^\theta P, w\rangle| + |\langle \overline{M_2}^\varphi P, w\rangle|\right) + 2\varepsilon^2 + 3\kappa(1+2\varepsilon).$$

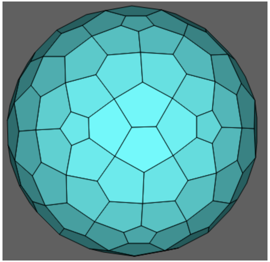
If $G^{\mathbb{Q}} > \max_{P \in \widetilde{\mathbf{P}}} H_P^{\mathbb{Q}}$ *then there does not exist a solution to Rupert's condition with*

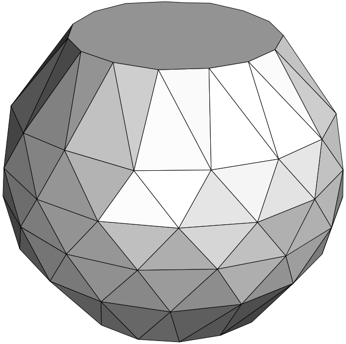
$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon].$$

Wishlist for a solid

1. Not Rupert
2. many symmetries
3. no mirrorsymmetry because of (A_ε) and (B_ε)
4. local theorem always applicable
5. pointsymmetry
6. not too many vertices







Definition of the Noperthedron

$$\mathcal{C}_{30} := \left\{ (-1)^\ell R_z\left(\frac{2\pi k}{15}\right) : k = 0, \ldots, 14; \ell = 0, 1 \right\}.$$

$$C_1 := \frac{1}{259375205}\begin{pmatrix}152024884\\0\\210152163\end{pmatrix}, \quad C_2 := \frac{1}{10^{10}}\begin{pmatrix}6632738028\\6106948881\\3980949609\end{pmatrix}, \quad C_3 := \frac{1}{10^{10}}\begin{pmatrix}8193990033\\5298215096\\1230614493\end{pmatrix}.$$

Note: $\|C_1\| = 1$ and $\frac{98}{100} < \|C_i\| < \frac{99}{100}$ for $i = 2, 3$.

Definition

Define the set of points $\mathbf{NOP} \subseteq \mathbb{R}^3$ by the action of $\mathcal{C}_{30}$ on C_1, C_2, C_3:

$$\mathbf{NOP} := \mathcal{C}_{30} \cdot C_1 \cup \mathcal{C}_{30} \cdot C_2 \cup \mathcal{C}_{30} \cdot C_3.$$

The *Noperthedron*, **NOP** has 90 vertices. **NOP** is pointsymmetric since $-\mathrm{Id} \in \mathcal{C}_{30}$.
Symmetries, e.g.: $M(\theta + 2\pi/15, \varphi) \cdot \mathbf{NOP} = M(\theta, \varphi) \cdot \mathbf{NOP} \Rightarrow \theta_1, \theta_2 \in [0, 2\pi/15)$.

Certificate of computer proof

ID	T	#C	ID1C	sp	$\overline{\theta}_{1,\min}$	$\overline{\theta}_{1,\max}$	$\overline{\varphi}_{1,\min}$	$\overline{\varphi}_{1,\max}$	$\overline{\theta}_{2,\min}$	$\overline{\theta}_{2,\max}$	$\overline{\varphi}_{2,\min}$	$\overline{\varphi}_{2,\max}$	$\overline{\alpha}_{\min}$	$\overline{\alpha}_{\max}$	P_1	P_2	P_3	Q_1	Q_2	Q_3	r	σ_Q	w_x	w_y	w_d	S
0	3	4	1	1	0	64..00	0	48..00	0	64..00	0	24..00	-24..00	24..00												
1	3	30	5	2	0	16..00	0	48..00	0	64..00	0	24..00	-24..00	24..00												
2	3	30	46..67	2	16..00	32..00	0	48..00	0	64..00	0	24..00	-24..00	24..00												
3	3	30	94..77	2	32..00	48..00	0	48..00	0	64..00	0	24..00	-24..00	24..00												
4	3	30	14..51	2	48..00	64..00	0	48..00	0	64..00	0	24..00	-24..00	24..00												
5	3	4	35	3	0	16..00	0	16..00	0	64..00	0	24..00	-24..00	24..00												
6	3	4	70..31	3	0	16..00	16..00	32..00	0	64..00	0	24..00	-24..00	24..00												
7	3	4	10..67	3	0	16..00	32..00	48..00	0	64..00	0	24..00	-24..00	24..00												
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
⋮ 29	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
91	1				0	80..00	0	80..00	80..00	16..00	80..60	16..20	-23..40	-22..80									53..73	15..64	16..45	39
92	1				0	80..00	80..00	16..00	0	80..00	0	80..60	-24..00	-23..40									98..92	35..15	10..33	37
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
245	2				0	20..00	0	20..00	0	20..00	0	20..40	-22..20	-22..80	30	31	38	79	80	87	-69	1				
246	1				0	20..00	0	20..00	0	20..00	20..40	40..80	-23..60	-22..20									71..05	19..88	20..37	39
247	1				0	20..00	0	20..00	0	20..00	20..40	40..80	-22..20	-22..80									71..05	19..88	20..37	39
248	2				0	20..00	0	20..00	20..00	40..00	0	20..40	-23..60	-22..20	30	31	38	79	80	87	-69	1				
249	2				0	20..00	0	20..00	20..00	40..00	0	20..40	-22..20	-22..80	30	31	38	79	80	87	-69	1				
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
18..44	1				48..00	64..00	46..00	48..00	48..00	64..00	22..80	24..00	22..80	24..00									33..40	-14..51	14..49	78

Computer proof statistics

- $\approx$ 18.000.000 global theorem applications
- $\approx$ 600.000 local theorem applications
- $\approx$ 3Gb uncompressed certificate ($\approx$ 150Mb compressed)
- $\approx$ 10h for creation of table (using floating points in R)
- $\approx$ 30h for verification in SageMath

Conclusion and open question

- What about the remaining 3 Archimedean solids? In particular the Rhombicosidodecahedron?
- Are there other ways to disprove the existence of local solutions?
- Is there a way to prove that a solid does not have Rupert's property without a huge case distinction?
- How to prove the conjectured Nieuwland numbers? $3\sqrt{2}/4$ for the Octahedron is open, also that the Dodecahedron and Icosahedron have $\nu \approx 1.0108$, a root of

$$P(x) = 2025x^8 - 11970x^6 + 17009x^4 - 9000x^2 + 2000.$$

- What is the link to duality?