

# Biomembranes and creative telescoping<sup>1</sup>

Seminar Algebra and Discrete Mathematics (Linz, Austria)

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<sup>1</sup>Joint work with [Alin Bostan](#) and [Thomas Yu](#).

# Motivating examples

- Recurrence for **Apery numbers**:

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \text{ satisfies } (n+1)^3 A_{n+1} = (17n^2 + 17n + 5)(2n+1)A_n - n^3 A_{n-1}$$

- Generating function of **moments**:

$$m_n = \int_0^1 x^n \frac{dx}{x(1-x)^3} \text{ satisfies } \sum_{k=0}^{\infty} m_k t^k = c {}_2F_1 \left( \begin{matrix} 1 & 4/3 \\ 8/3 \end{matrix}; t \right)$$

- **Surface area** a projection to  $\mathbb{R}^3$  of the **Clifford torus**:

$$\int_0^{2\pi} \int_0^{2\pi} \frac{(\sqrt{2} + \sin v)^2}{(1 + 2t(\sqrt{2} + \sin v) \cos u + t^2(3 + 2\sqrt{2} \sin v))^2} du dv$$

$$= \frac{4\sqrt{2}^2 (1-t^2)}{(t^2 - 6t + 1)^2} {}_2F_1 \left( \begin{matrix} 1/2 & 1/2 \\ 1 \end{matrix}; \frac{4t}{(1-t)^2} \right)$$

# Algorithmic proofs

$$A_n = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \text{ satisfies } (n+1)^3 A_{n+1} = (17n^2+17n+5)(2n+1)A_n - n^3 A_{n-1}:$$

$\underbrace{\binom{n}{k} \binom{n+k}{k}}_{=: a_{n;k}}$

[van der Poorten, 1978]:

Neither Cohen nor I had been able to prove (5) or (5') in the intervening 2 months. After a few days of fruitless effort the specific problem was mentioned to Don Zagier (Bonn), and with irritating speed he showed that indeed the sequence  $\{b'_n\}$  satisfies the recurrence (4). This more or less broke the dam and (5) and (5) were quickly conquered.

# Algorithmic proofs

$$A_n = \sum_{k=0}^n \binom{n^2}{k} \binom{n+k^2}{k} \text{ satisfies } (n+1)^3 A_{n+1} = (17n^2+17n+5)(2n+1)A_n - n^3 A_{n-1}:$$

> Zeilberger(a, n, k, N); finds in < 0.02 seconds:

$$L = (n+2)^3 N^2 - (17n^2 + 51n + 39)(2n+3)N + (n+1)^3 \text{ and}$$

$$C = (k^2 - 3 - 2k - 2n^2 - 6n - 4)k^4(16n+24) - (k - n - 1) = (k - n - 2);$$

with the property that  $(N - a_{n;k} := a_{n+1;k}$  and  $(K - a_{n;k} := a_{n;k+1})$ :

$$L \binom{n^2}{k} \binom{n+k^2}{k} = (K - 1) C \binom{n^2}{k} \binom{n+k^2}{k}:$$

Sum over  $k$  from 0 to  $n$  and conclude.



# Algorithmic proofs

$$m_n = \int_0^1 x^n \underbrace{\frac{1}{x(1-x)}}_{=: f_n(x)} dx \quad \text{satisfies} \quad \sum_{k=0}^n m_k t^k = {}_2F_1 \left( \begin{matrix} 1 \\ \frac{8}{3} \end{matrix}; t \right) = \frac{2^2}{15\Gamma(2/3)^3}.$$

> creative\_telescoping(f, n::Shift, x::Diff); finds in < 0.1 seconds:

$$L = (3n + 8)N - (3n + 4) \quad \text{and} \quad C(x) = 3x(x - 1);$$

with the property that  $(N - f_n(x)) = f_{n+1}(x)$ :

$$L \left( x^n \frac{1}{x(1-x)} \right) = @_x \left( C(x) x^n \frac{1}{x(1-x)} \right)$$

It follows that  $L \int_0^1 x^n \frac{1}{x(1-x)} dx = 0$  and hence  $(3n + 8)m_{n+1} = (3n + 4)m_n$ .

## Algorithmic proofs

$$\int_0^2 \int_0^2 \frac{(\sqrt{2} + \sin v) du dv}{(1 + 2t(\sqrt{2} + \sin v) \cos u + t^2(3 + 2\sqrt{2} \sin v))^2}$$

$$= \frac{4\sqrt{2} - 2 - t^2}{(t^2 - 6t + 1)^2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{4t}{(1-t)^2}\right)$$

# Algorithmic proofs

$$\frac{2(2^{\rho_{-2}} y^2 + 1)x \, dx dy}{2^{\rho_{-2}} t^2 x y^2 + 2^{\rho_{-2}} t x^2 y \quad t x^2 y^2 \quad 2^{\rho_{-2}} t^2 x \quad 2 t^2 x y + 2^{\rho_{-2}} t y + t x^2 \quad t y^2 \quad 2 y x + t^2}$$

$$= \frac{4^{\rho_{-2}} t^2}{(t^2 + 6t + 1)^2} {}_2F_1 \left( \begin{matrix} \frac{1}{2} \\ 1 \end{matrix}; \frac{4t}{(1-t)^2} \right) :$$

# Algorithmic proofs

$$\frac{2(2^{\rho-2}y^2 + 1)x \, dx dy}{2^{\rho-2}t^2xy^2 + 2^{\rho-2}tx^2y - tx^2y^2 - 2^{\rho-2}t^2x^2 - 2t^2xy + 2^{\rho-2}ty + tx^2 - ty^2 - 2yx + t^2}$$

$$= \frac{4^{\rho-2} \frac{1}{2} \frac{t^2}{6t+1} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{4t}{(1-t)^2}\right)}{(t^2 - 6t + 1)^2}$$

> FindCreativeTelescoping[F, fDer[x], Der[y]g, Der[t]]; finds in 10 seconds:

$$L = t^3t^2 - 1 - 9t^4 - 2t^2 + 1 - 3t^2 + 1^2 @_t^2 + 3t^2 + 1 - 729t^8 + 162t^6 - 192t^4 + 38t^2 - 1 @_t$$

$$+ 12t^3 - 324t^8 + 333t^6 + 51t^4 - 53t^2 + 1 ; \text{ and}$$

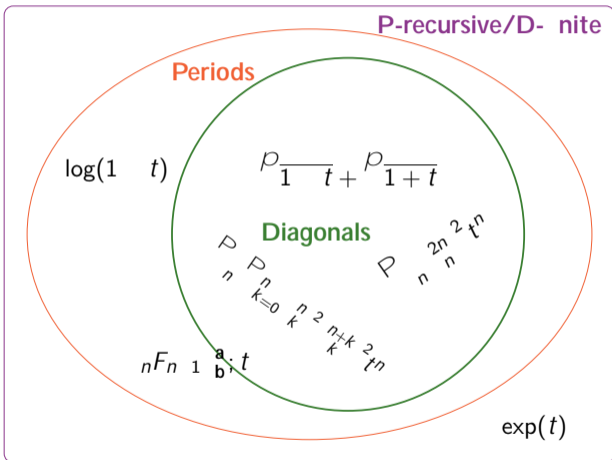
$C_1; C_2 \in \mathbb{Q}(x; y; t)$  with the property that:

$$L \cdot F = @_x C_1 + @_y C_2:$$

Therefore it follows that  $L \cdot F = 0$ . Solving  $Ly = 0$  we find the right-hand side.



# Definitions and interactions



A power series  $f(t) \in \mathbb{Q}[[t]]$  is **D-nite** if it satisfies a linear differential equation with polynomial coefficients:

$$p_n(t)f^{(n)}(t) + \dots + p_0(t)f(t) = 0:$$

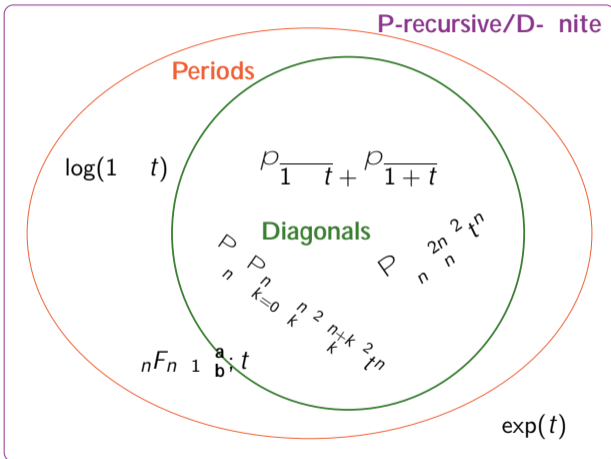
This equation can be rewritten:  $L f = 0$ ,

$$L = p_n(t)\partial_t^n + \dots + p_0(t) \in \mathbb{Q}[t][[\partial_t]]:$$

Let  $(\cdot)_n \equiv \prod_{i=0}^{n-1} (\cdot + i)$   
 Then  ${}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix}; t \right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} t^n$   
 satisfies

$$t(1-t)f''(t) + (c - (a+b+1)t)f'(t) - abf(t) = 0:$$

# Definitions and interactions



A sequence  $(u_n)_{n \geq 0}$  is **P-recursive**, if it satisfies a linear recurrence with polynomial coefficients:

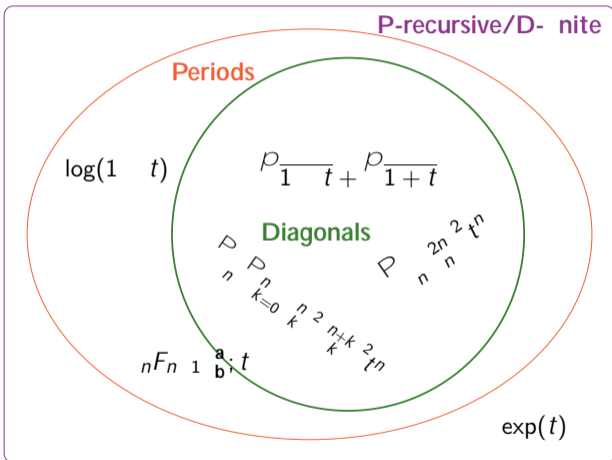
$$c_r(n)u_{n+r} + \dots + c_0(n)u_n = 0:$$

Let  $\binom{a}{n} = \frac{a(a-1)\dots(a-n+1)}{n!}$

Then  $u_n = \frac{\binom{a}{n} \binom{b}{n}}{(c)_n n!}$  satisfies

$$(c+n)(n+1)u_{n+1} - (a+n)(b+n)u_n = 0:$$

# Definitions and interactions



A power series  $f(t) \in \mathbb{Q}[[t]]$  is called a **Period function** if it is an integral of a rational function in  $t$  and  $x_1, \dots, x_n$  over a semi-algebraic set.

$$p(t) = 4 \int_0^1 \frac{1-t^2x^2}{1-x^2} dx$$

$$= 4 \int_0^1 \frac{dx dy}{1 - \frac{1-t^2x^2}{(1-x^2)y^2}} \text{ and}$$

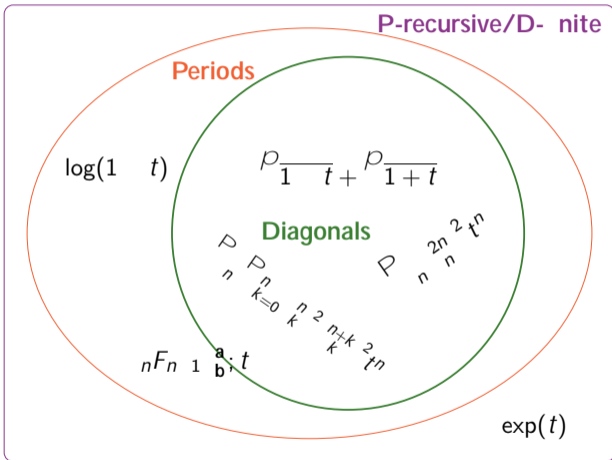
$$((t-t^3)^2 + (1-t^2)^2 + t) p = 0;$$

$$p(t) = 2 \frac{1}{2} t^2 - \frac{3}{32} t^4 \dots$$

**Andre-Bombieri-Katz's theorem:** A **Period function** is a G-function [André, 1989].

**Bombieri-Dwork conjecture:** Any G-function is a **Period function**.

# Definitions and interactions



A power series  $f(t) \in \mathbb{Q}[[t]] = \sum_{k=0}^{\infty} u_k t^k$  is called a **Diagonal** if there exists a rational function

$$R = \sum_{i_1, \dots, i_n=0}^{\infty} c_{i_1, \dots, i_n} x_1^{i_1} \dots x_n^{i_n} \in \mathbb{Q}(x_1, \dots, x_n)$$

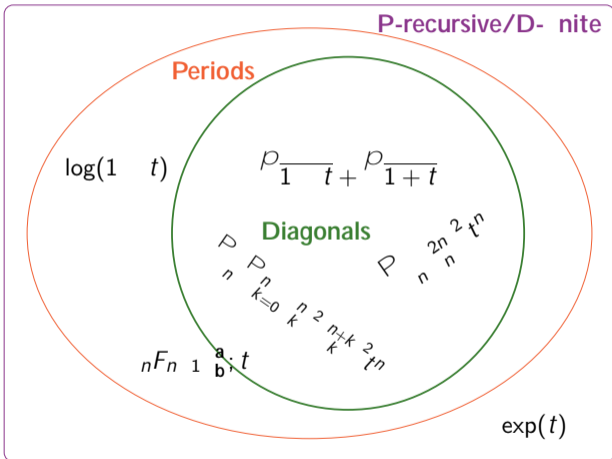
such that

$$f(t) = \text{Diag}(R) := \sum_{k=0}^{\infty} c_{k, \dots, k} t^k$$

Equivalently [Bostan, Lairez, Salvy 2017],  $(u_k)_{k=0}^{\infty}$  is a **multiple binomial sum**.

$$\text{Diag} \frac{1}{1-x-y} = \sum_{i,j=0}^{\infty} \binom{i+j}{j} x^i y^j = \sum_{k=0}^{\infty} \binom{2n}{n} t^k = \frac{1}{1-4t}$$

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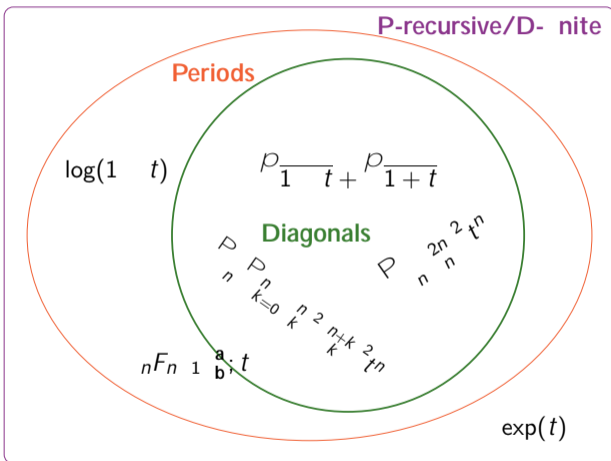
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Equivalently [Bostan, Lairez, Salvy 2017],  $(u_k)_{k \geq 0}$  is a **multiple binomial sum**.

$$\text{Diag} \frac{1}{1-x-y} = [x^1] \frac{1}{x-1} \frac{1}{x-t=x} = \frac{1}{2} \sum_{|j_x|=1} \frac{dx}{x-x^2-t} = (1-4t)^{-\frac{1}{2}}$$

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**Christol's conjecture:** A convergent **D- nite** power series in  $\mathbb{Z}[[t]]$  is a **Diagonal**.

# Principle of Creative Telescoping

- Goal: Given a **Period function** or **Diagonal**, find an annihilating ODE.

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- More precisely: Given  $R \in \mathbb{Q}(x_1; \dots; x_n; t)$  and a closed cycle  $C^n$ , find

$$L = p_n(t) \partial_t^n + \dots + p_0(t) \in \mathbb{Q}[t][\partial_t]; \quad \text{such that} \quad L R dx = 0:$$



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- Note:  $\partial_{x_i} C dx = \dots$ ,  $C dx = \dots$ ,  $C dx = 0$  for any rational function  $C \in \mathbb{Q}(\mathbf{x}; t)$ .
- So we need to find

$$L \in \mathbb{Q}[t][\partial_t]; \quad \text{and} \quad C_1; \dots; C_n \in \mathbb{Q}(x_1; \dots; x_n; t); \quad \text{such that}$$

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## Principle of Creative Telescoping

$$\sum_{k=0}^n p_k(t) \frac{d^k R}{dt^k} = \frac{\partial}{\partial x_1} C_1 + \dots + \frac{\partial}{\partial x_n} C_n \quad \sum_{k=0}^n p_k(t) \frac{\partial^k}{\partial t^k} R dx = 0:$$

The **telescoper** and **certificates** always exist and can be found **algorithmically**.

# The Almkvist-Zeilberger algorithm [1990] *"I could never resist a de nite integral."*

**Input:** A hyperexponential function  $H(t; x)$ , i.e.  $@_t H = H$  and  $@_x H = H \cdot Q(t; x)$ .

**Output:** A linear differential operator  $P(t; @_t) \in \mathbb{Q}[t][@_t]$  and  $G(t; x) \in \mathbb{Q}(t; x)$ , s.t.

$$P H = @_x (G H):$$

**Algorithm:** Let  $L = \mathbb{Q}(t)$ . For  $r = 0; 1; 2; \dots$  do:

- 1 Compute  $a(t; x) = @_x H = H$  and  $b_k(t; x) = @_t^k H = H$  for  $k = 0; \dots; r$ .
- 2 Decide whether the (ordinary, linear, inhomogeneous, parametrized) diff. equation

$$@_x G + a(t; x)G = \sum_{k=0}^r c_k(t)b_k(t; x)$$

has a rational solution  $G \in L(x)$  for some  $c_0(t); \dots; c_r(t) \in L$  not all zero.

- 3 If found solution in (2), return  $P = \sum_{k=0}^r c_k @_t^k$  and  $G$ ; else increase  $r$  and repeat.

# Some history of Creative Telescoping

- Indefinite integration/summation and working examples

- Sums: [Bernoulli, Fasenmyer, Gosper,...]

- Integrals: [Legendre, Ostrogradsky, Hermite, Picard, Manin, Griffiths, Feynman, ...]

$$\frac{P(x)}{Q(x)} dx = \frac{P_1(x)}{Q_1(x)} + \frac{P_2(x)}{Q_2(x)} dx$$

$$\sum_{k=1}^n \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$$

$$\int_0^1 \frac{dx}{x(1-x)(1-xt)} = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; t\right)$$

- Algorithmic Creative Telescoping (algorithmic definite summation&integration):

- 1G: brutal elimination: [Fasenmyer, 1947], [Zeilberger, 1990], [Takayama, 1990]

- 2G: rational solutions of linear ODEs: [Zeilberger, 1990], [Almkvist, Zeilberger, 1990], [Chyzak, 2000], [Koutschan, 2010]

- 3G: 2G + linear algebra + bounds: [Apagodu, Zeilberger, 2005], [Koutschan 2010], [Chen, Kauers 2012], [Chen, Kauers, Koutschan 2014]

- 4G: based on (Hermite- and generalized Griffiths-Dwork) reduction [Bostan, Chen, Chyzak, Kauers, Koutschan, Li, Lairez, Salvy, Singer,...]

## Creative Telescoping and de Rham cohomology

*"the certificate is not needed, its existence and regularity are sufficient."*

- Let  $L = \mathbb{Q}(t)$ ,  $f \in L[x_0; \dots; x_n] = L[x]$  and  $C^n$  a closed  $n$ -cycle.
- Denote by  $L[x; 1=f]_p = \{F \in L[x; 1=f] : F(x) = pF(x); p \in \mathbb{Q}(t)\}$ .
- We wish to compute the differential equation satisfied by

$$F(t; x_0; \dots; x_n) dx; \text{ where } F = a \cdot f' \in L[x; 1=f]_{n-1}$$

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- We wish to compute the differential equation satisfied by

$$F(t; x_0; \dots; x_n) dx; \text{ where } F = a \cdot f^{-1} \in L[x; 1=f]_{n-1}$$

- Therefore we wish to find a non-trivial element in

$$H_f^{\text{pr}} := L[x; 1=f]_{n-1} / D_f; \text{ where } D_f := \text{span}_{\mathbb{Q}}(f @_{x_i} C : C \in L[x; 1=f]_{n-1})$$

- Generalized Griffiths-Dwork Reduction:  $F \notin [F]$ , s.t.  $F dx = 0 \iff [F] = 0$ .

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Theorem [Griffiths 1969, Bostan, Lairez, Salvy 2013, Lairez 2016]

Assume that  $L[x] = \langle h @_{x_0} f; \dots; @_{x_n} f \rangle$  is finite-dimensional over  $L$ . Then  $H_f^{\text{pr}}$  is finitely generated over  $L$ . Moreover the *Generalized Griffiths-Dwork Reduction* can be used to compute the (minimal regular) **telescoper**.

## Issues with singularities: non-regular certificates

*\the certi cate is not needed, its existence and regularity are sufficient."*

- The following example originates in [Picard, 1899]: Let  $P_t(u) = u^3 + t$ , then

$$F = \frac{x \ y}{z^2 \ P_t(x)P_t(y)}$$

$$= @_x \frac{2P_t(x)}{(x \ y)(z^2 \ P_t(x)P_t(y))} + @_y \frac{2P_t(y)}{(x \ y)(z^2 \ P_t(x)P_t(y))} + @_z \frac{3(x^2 + y^2)z}{(x \ y)(z^2 \ P_t(x)P_t(y))};$$



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- So one has  $F = @_x C_1 + @_y C_2 + @_z C_3$ , however:

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$$F \, dx \, dy \, dz \notin 0 \quad \text{for some} \quad \mathbb{C}^3:$$

- Conclusion: Certificates are important.  
A certificate is called **regular** if it has no other poles than  $F$ .

# Motivation and Introduction

"Why do all humans have the same biconcave shaped red blood cells?"

- Canham model predicts shape of biomembranes like blood cells [Canham, 1970].
- The model asks to minimize the Willmore energy

$$W(S) := \int_S H^2 dA; \quad (H \text{ is the mean curvature})$$

over orientable closed surfaces  $S \subset \mathbb{R}^3$  with genus  $g$ , area  $A_0$  and volume  $V_0$ .

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- [Willmore, 1965]: For a torus  $T = T(R; r)$  the Willmore energy is:

$$W(T) = \frac{p}{r} \frac{2R^2}{R^2 - r^2} \quad \text{minimal for } R=r = \sqrt{\frac{p}{2}}:$$

Theorem (Willmore 1964 (conjectured); Marques, Neves, 2014)

Across all closed surfaces  $\mathbb{R}^3$  of genus  $g \geq 1$  the Willmore energy is minimal for  $T^g \sqrt{\frac{p}{2}}$ .

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over orientable closed surfaces  $S \subset \mathbb{R}^3$  with genus  $g$ , area  $A_0$  and volume  $V_0$ .

- [Willmore, 1965]: For a torus  $T = T(R; r)$  the Willmore energy is:

$$W(T) = \frac{p}{r} \frac{2R^2}{R^2 - r^2} \quad \text{minimal for } R=r = \sqrt{\frac{p}{2}}:$$

Theorem (Willmore 1964 (conjectured); Marques, Neves, 2014)

Across all closed surfaces  $S \subset \mathbb{R}^3$  of genus  $g \geq 1$  the Willmore energy is minimal for  $T \sqrt{\frac{p}{2}}$ .

- $W(S)$  is invariant under Möbius transformations  $\Rightarrow$  no uniqueness of the shape.

# [Yu, Chen, 2021]: All projections of the (Cli ord) torus

- The Cli ord torus CT is defined as the following set in  $\mathbb{S}^3$ :

$$CT := \{ [\cos u; \sin u; \cos v; \sin v]^T \mid u, v \in [0; 2\pi] \} \subset \mathbb{R}^4$$

- The torus with minor radius 1 and major radius  $R > 1$ :

$$T_R := \{ [(R + \cos v) \cos u; (R + \cos v) \sin u; \sin v]^T \mid u, v \in [0; 2\pi] \} \subset \mathbb{R}^3$$

- $\text{inv}_{(x;y;z)}$  is the inversion map about the unit sphere centered at  $(x; y; z) \in \mathbb{R}^3$ .

# [Yu, Chen, 2021]: All projections of the (Cli ord) torus

- The Cli ord torus CT is defined as the following set in  $\mathbb{S}^3$ :

$$CT := \{ [\cos u; \sin u; \cos v; \sin v]^T \in \mathbb{P}^1 \times \mathbb{P}^1 : u, v \in [0; 2\pi] \} \subset \mathbb{R}^4$$

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$$T_R := \{ [(R + \cos v) \cos u; (R + \cos v) \sin u; \sin v]^T : u, v \in [0; 2\pi] \} \subset \mathbb{R}^3$$

- $\text{inv}_{(x;y;z)}$  is the inversion map about the unit sphere centered at  $(x; y; z) \in \mathbb{R}^3$ .
- The set of all shapes of stereographic projections of CT to  $\mathbb{R}^3$  is parameterized by

$$\{ \text{inv}_{(t;0;0)}(\mathbb{P}^1 \times \mathbb{P}^1) : t \in [0; \infty) \}$$

$$W(\text{inv}_{(x;y;z)}(T)) = W(T) = \int_T H^2 dA = 2 \int_T \dots$$



## [Willmore, 1965] and [Marques, Neves, 2014]

[Marques, Neves, 2014] Let  $S^3$  be an embedded closed surface of genus 1. Then  $W(\Sigma) = 2\pi^2$  and the equality holds if and only if  $\Sigma$  is the Clifford torus up to conformal transformations of  $S^3$ .

# Uniqueness with prescribed isoperimetric ratio

- In Canham's model, instead of  $A_0$  and  $V_0$  rather prescribe the isoperimetric ratio

$$\rho_0 := \frac{1}{2} \sqrt{\frac{6V_0}{A_0}} \in (0; 1]:$$

## Question

Is the minimizer of  $W(S)$  with prescribed genus  $g$  and isoperimetric ratio  $\rho_0$  unique?

# Uniqueness with prescribed isoperimetric ratio

- In Canham's model, instead of  $A_0$  and  $V_0$  rather prescribe the isoperimetric ratio

$$\rho_0 := \frac{3\sqrt{6V_0}}{A_0} \in (0; 1]:$$

## Question

Is the minimizer of  $W(S)$  with prescribed genus  $g$  and isoperimetric ratio  $\rho_0$  unique?

Theorem (Yu, Chen, 22; Melczer, Mezzarobba, 22; Bostan, Y., 22)

The shape of the projection of the Clifford torus to  $\mathbb{R}^3$  is uniquely determined by  $\rho_0$ . Thus, if  $g = 1$  and  $\rho_0 \in (2^{-3/2}, 1)$  then Canham's model has a unique solution.

## Summary of [Yu, Chen, 22] and [Melczer, Mezzarobba, 22]

- Let  $(S) := \frac{1-6^p}{6V(S)} \overline{A(S)}^p$  and  $(0; 1]$ ; and  $(S) := 3 = (2^{5-4} \overline{A(S)}^p)$   $(0; 1]$ . Define
 
$$\text{Iso: } [0, \overline{A(S)}^p] \rightarrow [0, 1];$$

$$t \mapsto \frac{1}{7!} (\text{inv}_{(t;0;0)}(T^{\overline{A(S)}^p}))^3$$
- $\overline{A(S)}^p$  is the surface area and  $\overline{V(S)}^p$  is the volume of  $\text{inv}_{(t;0;0)}(T^{\overline{A(S)}^p})$ .

## Summary of [Yu, Chen, 22] and [Melczer, Mezzarobba, 22]

- Let  $(S) := \frac{1}{6} \frac{d}{dt} \frac{V(S)}{A(S)} \Big|_{t=0}$ ; and  $iso := \frac{3}{2} (2^{5-4} P^{-}) \Big|_{t=0}$ . Define

$$Iso: [Q^{\frac{P}{2}} \bar{1}]! \quad [; 1);$$

$$t \cdot 7! \cdot (\text{inv}_{(t;0;0)}(T^{\frac{P}{2}}))^{-3}$$

- $\frac{P}{2} A(t^2)$  is the surface area and  $\frac{P}{2} V(t^2)$  is the volume of  $\text{inv}_{(t;0;0)}(T^{\frac{P}{2}})$ .
- [Yu, Chen, 22]: Enough to show:  $Iso(t)$  is strictly increasing. Moreover,

$$\frac{V(t^2)A(t^2)}{2^4} \frac{d}{dt} \ln(Iso(t)^2) = 72t + 1932t^3 + 31248t^5 + \dots =: \sum_{n=0}^{\infty} a_n t^n$$

is a **D-nite** function. Enough to show  $a_n > 0$  for all  $n \geq 0$ .

# Summary of [Yu, Chen, 22] and [Melczer, Mezzarobba, 22]

- Let  $(S) := \mathbb{P}_3^1 \overline{6V(S)} = \mathbb{P}_2 \overline{A(S)} \in (0; 1]$ ; and  $:= 3 = (2^{5=4} \mathbb{P}^-) \in (0; 712]$ . Define

$$\text{Iso}: [0, \sqrt{2} - 1] \rightarrow [0; 1];$$

$$t \mapsto 7! \cdot (\text{inv}_{(t;0;0)}(T^{\mathbb{P}_2}))^3$$

- $\sqrt{2} - 2A(t^2)$  is the surface area and  $\sqrt{2} - 2V(t^2)$  is the volume of  $\text{inv}_{(t;0;0)}(T^{\mathbb{P}_2})$ .
- [Yu, Chen, 22]: Enough to show: Iso(t) is strictly increasing. Moreover,

$$\frac{V(t^2)A(t^2)}{2^4} \frac{d}{dt} \ln(\text{Iso}(t)^2) = 72t + 1932t^3 + 31248t^5 + \dots =: \sum_{n=0}^{\infty} a_n t^n$$

is a **D-nite** function. Enough to show  $a_n > 0$  for all  $n \geq 0$ .

- [Melczer, Mezzarobba, 22]: Rigorous asymptotics & error bounds  $a_n > 0$ .

Therefore, Iso(t) is increasing.

# Closed form solution

## Proposition (Bostan, Y., 2022)

The surface area  $\sqrt[2]{2} A(t^2)$  and volume  $\sqrt[2]{2} V(t^2)$  of  $\text{inv}_{(t;0;0)}(\mathbb{TP}^2)$  are given by

$$A(t) = \frac{4(1-t^2)}{(t^2-6t+1)^2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{4t}{(1-t)^2}\right);$$

$$V(t) = \frac{2(1-t)^3}{(t^2-6t+1)^3} {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; \frac{4t}{(1-t)^2}\right);$$

## Corollary

The function  $\text{Iso}(t)^2 = 36 \frac{V(t^2)^2}{A(t^2)^3}$  is increasing on  $(0; \sqrt[2]{2}-1)$ .

# Proof of closed-form for $V(t)$

- Let  $Q(u; v; r; t) = 1 + 2t(\sqrt{2} + r \sin v) \cos u + t^2(2 + r^2 + 2\sqrt{2}r \sin v)t^2$ . Then

$$\begin{aligned}
 \int_0^1 \int_0^1 \int_0^1 \frac{r \sqrt{2} + r^2 \sin(v)}{Q(u; v; r; t)^3} du dv dr \\
 = \int_0^1 \int_{|x|=|y|=1} F(x; y; r; t) dx dy dr = 2 + 48t^2 + \frac{1269}{2}t^4 + \dots
 \end{aligned}$$

for some  $F(x; y; r; t) \in \mathbb{Q}(x; y; r; t; \sqrt{2})$ . Thus  $V(t)$  is a **period function**.



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$$\begin{aligned}
 \int_0^1 \int_0^{2\pi} \int_0^1 \frac{r^2 \sqrt{2} + r^2 \sin(v)}{Q(u; v; r; t)^3} du dv dr \\
 = \int_0^1 \int_{|x|=|y|=1} F(x; y; r; t) dx dy dr = 2 + 48t^2 + \frac{1269}{2}t^4 + \dots
 \end{aligned}$$

for some  $F(x; y; r; t) \in \mathbb{Q}(x; y; r; t; \sqrt{2})$ . Thus  $V(t)$  is a **period function**.

- First try: Use creative telescoping on the triple integral:

> FindCreativeTelescoping[F, {Der[x], Der[y], Der[r]}, g, Der[t]];

finds  $C_1; C_2; C_3 \in \mathbb{Q}(x; y; r; t)$  such that  $F = \partial_x C_1 + \partial_y C_2 + \partial_r C_3$ .

## Proof of closed-form for $V(t)$

- Let  $Q(u; v; r; t) = 1 + 2t(\sqrt{2} + r \sin v) \cos u + t^2(2 + r^2 + 2\sqrt{2}r \sin v)t^2$ . Then

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for some  $F(x; y; r; t) \in \mathbb{Q}(x; y; r; t; \sqrt{2})$ . Thus  $V(t)$  is a **period function**.

- Second try: Find a closed form for  $\int F dx dy$  and integrate  $dr$  "by hand".

`> FindCreativeTelescoping[F, {Der[x], Der[y]}, g, Der[t]];`

returns  $L \in \mathbb{Q}[r; t][\sqrt{2}]$  and  $C_1; C_2 \in \mathbb{Q}(x; y; r; t)$  s.t.  $L F = \frac{\partial}{\partial x} C_1 + \frac{\partial}{\partial y} C_2$ .

- The common denominator of  $C_1$  and  $C_2$  is

$$\text{denom}(F) = x y (1 + 2\sqrt{2}y - y^2) H(t; r):$$

## Proof of closed-form for $V(t)$

- Let  $Q(u; v; r; t) = 1 + 2t(\sqrt{2} + r \sin v) \cos u + t^2(2 + r^2 + 2\sqrt{2}r \sin v)t^2$ . Then

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 = \int_0^1 \int_{|x|=|y|=1} F(x; y; r; t) dx dy dr = 2 + 48t^2 + \frac{1269}{2}t^4 + \dots
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for some  $F(x; y; r; t) \in \mathbb{Q}(x; y; r; t; \sqrt{2})$ . Thus  $V(t)$  is a **period function**.

- Second try: Find a closed form for  $\int F dx dy$  and integrate  $dr$  by hand".

`> FindCreativeTelescoping[F, f Der[x], Der[y] g, Der[t]];`

returns  $L \in \mathbb{Q}[r; t][\sqrt{2}]$  and  $C_1; C_2 \in \mathbb{Q}(x; y; r; t)$  s.t.  $L F = \frac{\partial}{\partial x} C_1 + \frac{\partial}{\partial y} C_2$ .

- The common denominator of  $C_1$  and  $C_2$  has

$$\text{denom}(F) = x y (1 + 2\sqrt{2}y - y^2) H(t; r) \quad = ; :$$

$$P \int_{\bar{2}}^2 V(t^2) = \int_0^1 \underbrace{F(x; y; r; t) dx dy}_{=: G(r; t)} dr:$$

$G(r; t)$  satisfies  $(P_2(r; t) \frac{\partial}{\partial r} + P_1(r; t) \frac{\partial}{\partial t} + P_0(r; t)) G(r; t) = 0$ . Then:

$$G(r; t) = Q_1 {}_2F_1 \left( \frac{3}{2}; \frac{3}{2}; 1 \right) + Q_2 {}_2F_1 \left( \frac{1}{2}; \frac{3}{2}; 2 \right);$$

for some (explicit)  $Q_1; Q_2; \frac{\partial}{\partial r}; \frac{\partial}{\partial t} \in Q(r; t)$ . Then we also find:

$$\int_0^s G(r; t) dr = \frac{2(s-t^2)^3}{(2-s)t^4} {}_2F_1 \left( \frac{3}{2}; \frac{3}{2}; 1 \right); \frac{4t^2 s}{(1-t^2(2-s))^2} :$$

Finally:  $P \int_{\bar{2}}^2 V(t^2) = \int_0^1 G(r; t) dr$ , so sets = 1 above.

# Iso is bijective

## Proposition

Let

$$A(t) = \frac{4 - t^2}{(t^2 - 6t + 1)^2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{4t}{(1-t)^2}\right);$$

$$V(t) = \frac{2(1-t)^3}{(t^2 - 6t + 1)^3} {}_2F_1\left(\frac{3}{2}, \frac{3}{2}; 1; \frac{4t}{(1-t)^2}\right);$$

Then  $\text{Iso}(t)^2 = 36 \frac{V(t^2)^2}{A(t^2)^3}$  is increasing on  $(0; \sqrt{2} - 1)$ .

# Iso is bijective

We need to show that

$$z \mapsto \frac{{}_2F_1\left(\begin{matrix} h \\ 1 \end{matrix}; \frac{4z}{(1-z)^2}\right)}{{}_2F_1\left(\begin{matrix} \frac{1}{2} \\ 1 \end{matrix}; \frac{4z}{(1-z)^2}\right)}$$

is increasing on  $[0; 3/2)$ .

# Iso is bijective

We need to show that

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is increasing on  $[0; \frac{3}{2}]$ . Let  $x = 4z/(1-z)^2$ , then it remains to show that

$$h: x \mapsto \frac{{}_2F_1\left(\frac{3}{2}; \frac{3}{2}; x\right)}{{}_2F_1\left(\frac{1}{2}; \frac{1}{2}; x\right)}$$

is increasing on  $[0; 1)$ .

# Iso is bijective

We need to show that

$$z \mapsto \frac{{}_2F_1\left(\frac{3}{2}, \frac{3}{2}; \frac{4z}{(1-z)^2}; i\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{4z}{(1-z)^2}; i\right)} = \frac{1-z}{1+z}^3$$

is increasing on  $[0; \frac{2}{3}]$ . Let  $x = 4z = (1-z)^2$ , then it remains to show that

$$h: x \mapsto \frac{{}_2F_1\left(\frac{3}{2}, \frac{3}{2}; x; i\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; x; i\right)} = (x+1)^{-3/2}$$

is increasing on  $[0; 1]$ . Observe:  $h$  can be written as  $h(x) = g(x)^2 = f(x)^3$ , where

$$g(x) = \frac{{}_2F_1\left(\frac{3}{2}, \frac{3}{2}; x; i\right)}{(x+1)^{3/2}} \quad \text{and} \quad f(x) = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; x; i\right)}{(x+1)^{1/2}}$$

To show:  $g(x)$  is increasing and  $f(x)$  is decreasing on  $(0; 1)$ .



# Iso is bijective

## Proposition

Let  $a \in \mathbb{R}$  and let  $w_a : [0; 1] \rightarrow \mathbb{R}$  be defined by

$$w_a(x) = {}_2F_1 \left( \begin{matrix} a \\ 1 \end{matrix}; x \mid (x+1)^{-a} \right)$$

Then  $w_a$  is: decreasing if  $0 < a < 1$ ; increasing if  $a > 1$ ; constant if  $a \in \{0, 1\}$ .

Clearly,  $g(x) = w_{3=2}(x)$  and  $f(x) = w_{1=2}(x)$ .

# Iso is bijective

## Proposition

Let  $a \in \mathbb{R}$  and let  $w_a : [0; 1] \rightarrow \mathbb{R}$  be defined by

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Then  $w_a$  is: decreasing if  $0 < a < 1$ ; increasing if  $a > 1$ ; constant if  $a \in \{0, 1\}$ .

## Proof.

$$\frac{w_a'(x)}{w_a(x)} = \frac{-(a+1)(x+1)^{-a-1}}{(1-x)^{2a}} = {}_2F_1 \left( \begin{matrix} a+1 \\ 2 \end{matrix}; x \mid \dots \right) \quad \square$$

Clearly,  $g(x) = w_{3/2}(x)$  and  $f(x) = w_{1/2}(x)$ .

# The general case $R > 1$

Recall:

$$T_R := \left\{ [(R + \cos v) \cos u; (R + \cos v) \sin u; \sin v]^T : u, v \in [0; 2\pi] \times [0; \pi] \right\} \subset \mathbb{R}^3; \text{ and}$$

$\text{inv}_{(x;y;z)}$  is the inversion about the unit sphere centered at  $(x; y; z)$ .

## Question

Are there closed formulas for the volume and surface area of  $\text{inv}_{(x;y;z)}(T_R)$  for any  $R$ ?  
Is  $\text{Iso}_R(t)$  increasing in  $t$  for any  $R > 1$ ?

# Computing the isoperimetric ratio

## Theorem (Bostan, Yu, Y., 2023)

The surface area  $A_R(t^2)R^{-2}$  and volume  $V_R(t^2)R^{-2}$  of  $\text{inv}_{(t;0;0)}\left(\frac{T_R}{R^2-1}\right)$  are given by

$$A_R(t) = \frac{4(1 - (R^2 - 1)t^2)}{(1 - 2(R^2 + 1)t + (R^2 - 1)t^2)^2} {}_2F_1\left[\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix}; \frac{4t}{(1 - (R^2 - 1)t)^2}\right];$$

$$V_R(t) = \frac{2(1 - (R^2 - 1)t)^3}{(1 - 2(R^2 + 1)t + (R^2 - 1)t^2)^2} {}_3F_2\left[\begin{matrix} \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} \end{matrix}; \frac{4t}{(1 - (R^2 - 1)t)^2}\right];$$

## Corollary

For  $R > 1$  the function  $\text{Iso}_R^2(t^2) = 36 \frac{V_R(t^2)^2}{A_R(t^2)^3}$  is increasing on  $[0; (R + 1)^{-1}]$ .

## Theorem

For  $R > 1$  the function  $I_{\mathbb{R}}^2(t^2) = 36 \frac{V_{\mathbb{R}}(t^2)^2}{A_{\mathbb{R}}(t^2)^3}$  is increasing on  $\mathbb{R}^2(0; (R+1)^{-1})$ , with

$$A_{\mathbb{R}}(t) = \frac{4(1 - (R^2 - 1)t^2)}{(1 - 2(R^2 + 1)t + (R^2 - 1)t^2)^2} \quad {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; \frac{4t}{(1 - (R^2 - 1)t^2)} \right); \quad \#$$

$$V_{\mathbb{R}}(t) = \frac{2(1 - (R^2 - 1)t^2)^3}{(1 - 2(R^2 + 1)t + (R^2 - 1)t^2)^2} \quad {}_3F_2 \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2R^2 - 4} + 1; \frac{4t}{(1 - (R^2 - 1)t^2)} \right); \quad \#$$

First perform the substitution  $x = 4t^2 = (1 - (R^2 - 1)t^2)^2$ . It remains to show that:

$$h(x) := {}_3F_2 \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2(R^2 - 2)} + 1; \frac{3}{2(R^2 - 2)} \right); x \quad {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; x \right)^3 (1 + (R^2 - 1)x)^{-3/2}$$

is increasing on  $\mathbb{R}^2(0; 1)$  for all  $R > 1$ .

$$h(x) := {}_3F_2 \left( \begin{matrix} 2 \\ 3/2 \end{matrix} ; x \right) = {}_2F_1 \left( \begin{matrix} 3/2 \\ 1 \end{matrix} ; x \right) \frac{1}{(1 + (R^2 - 1)x)^{3/2}}$$

is increasing on  $x \in (0; 1)$  for all  $R > 1$ .

$$h(x) := {}_3F_2 \left( \begin{matrix} 2 \\ 3 \end{matrix} ; x \right) {}_2F_1 \left( \begin{matrix} \frac{1}{2} \\ 1 \end{matrix} ; x \right) (1 + (R^2 - 1)x)^{3/2}$$

is increasing on  $(0; 1)$  for all  $R > 1$ . Note that  $h(x) = g(x)^2 f(x)^3$ , where

$$g(x) := \frac{{}_3F_2 \left( \begin{matrix} \frac{3}{2} \\ 1 \end{matrix} ; x \right)}{(1+x)^{3/2} (1+(R^2-1)x)^{3/2}} \quad \text{and} \quad f(x) := {}_2F_1 \left( \begin{matrix} \frac{1}{2} \\ 1 \end{matrix} ; x \right) (x+1)^{-1/2}$$

$$h(x) := {}_3F_2 \left( \begin{matrix} 2 \\ 3 \end{matrix}; \begin{matrix} \frac{3}{2} \\ 1 \end{matrix}; x \right) \frac{{}_3F_2 \left( \begin{matrix} 3 \\ 2 \end{matrix}; \frac{3}{2(R^2-2)} + 1 \right)}{{}_3F_2 \left( \begin{matrix} 3 \\ 2 \end{matrix}; \frac{3}{2(R^2-2)} \right)}; x^5 \quad {}_2F_1 \left( \begin{matrix} \frac{1}{2} \\ 1 \end{matrix}; x \right)^3 (1 + (R^2 - 1)x)^{3-2}$$

is increasing on  $(0; 1)$  for all  $R > 1$ . Note that  $h(x) = g(x)^2 = f(x)^3$ , where

$$g(x) := \frac{{}_3F_2 \left( \begin{matrix} 3 \\ 2 \end{matrix}; \frac{3}{2} \\ 1 \frac{3}{2} \frac{3}{2(R^2-2)} + 1 \right); x}{(1+x)^{3-4} (1+(R^2-1)x)^{3-4}} \quad \text{and} \quad f(x) := {}_2F_1 \left( \begin{matrix} \frac{1}{2} \\ 1 \end{matrix}; x \right) (x+1)^{1-2};$$

We already saw  $f(x)$  is decreasing.



$$h(x) := {}_3F_2 \left( \begin{matrix} 2 \\ 3 \\ 2 \end{matrix} \middle| \begin{matrix} 3 \\ 2 \\ 2(R^2 - 2) + 1 \end{matrix} ; x \right) {}_5F_1 \left( \begin{matrix} 3_2 \\ 1 \\ 2 \end{matrix} \middle| \begin{matrix} 1 \\ 2 \\ 1 \\ 2 \end{matrix} ; x \right) (1 + (R^2 - 1)x)^{3-2}$$

is increasing on  $(0; 1)$  for all  $R > 1$ . Note that  $h(x) = g(x)^2 = f(x)^3$ , where

$$g(x) := \frac{{}_3F_2 \left( \begin{matrix} 3 \\ 2 \\ 2 \end{matrix} \middle| \begin{matrix} 3 \\ 2 \\ 2(R^2 - 2) + 1 \end{matrix} ; x \right)}{(1+x)^{3-4} (1+(R^2-1)x)^{3-4}} \quad \text{and} \quad f(x) := {}_2F_1 \left( \begin{matrix} 1 \\ 2 \\ 1 \end{matrix} \middle| \begin{matrix} 1 \\ 2 \end{matrix} ; x \right) (x+1)^{1-2}$$

We already saw  $f(x)$  is decreasing. For  $g(x)$  it holds that:

$$\frac{4 g'(x)}{3 (1+x)^2 (R^2-1)} = \sum_{n=0}^{\infty} u_n(R) x^n; \quad \text{and}$$

$u_{n+1}(R) = u_n(R) = (2n-1)(2n+1) p_{n+1}(R) = (4(n+2)(n+1) p_n(R))$ ,  $u_0(R) = 1$ , where

$$p_n(R) := 4(R^4 + 4R^2 - 4)n^3 + 6(R^4 + R^2 - 2)n^2 + (2R^4 - 13R^2 + 10)n - 3R^2 + 3 > 0:$$

## Summary and conclusion

- Creative Telescoping is a powerful tool for dealing with **Period functions**.
- Implemented versions of Creative Telescoping exist (both 2G and 4G). They are useful in practice and can solve non-trivial problems.
- The surface area and volume of any stereographic projection to  $\mathbb{R}^3$  of the Clifford torus can be expressed in terms of **hypergeometric functions**.
- The Canham model in genus 1 has a unique solution when  $\frac{3}{0} \geq \frac{3}{2^{5-4}} \frac{1}{2}; 1$ .