

# Biomembranes and creative telescoping<sup>1</sup>

Seminar Algebra and Discrete Mathematics (Linz, Austria)

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<sup>1</sup>Joint work with [Alin Bostan](#) and [Thomas Yu](#).

# Motivating examples

- Recurrence for **Apéry numbers**:

$$A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \text{ satisfies } (n+1)^3 A_{n+1} = (17n^2 + 17n + 5)(2n+1)A_n - n^3 A_{n-1}.$$

- Generating function of **moments**:

$$m_n = \int_0^1 x^n \cdot \sqrt[3]{x(1-x)} dx \text{ satisfies } \sum_{k \geq 0} m_k t^k = c \cdot {}_2F_1 \left[ \begin{matrix} 1 & \frac{4}{3} \\ \frac{8}{3} \end{matrix}; t \right].$$

- **Surface area** a projection to  $\mathbb{R}^3$  of the **Clifford torus**:

$$\int_0^{2\pi} \int_0^{2\pi} \frac{(\sqrt{2} + \sin v) du dv}{(1 + 2t(\sqrt{2} + \sin v) \cos u + t^2(3 + 2\sqrt{2} \sin v))^2} = \frac{4\sqrt{2}\pi^2 (1-t^2)}{(t^2 - 6t + 1)^2} {}_2F_1 \left[ \begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 \end{matrix}; \frac{4t}{(1-t)^2} \right]$$

# Algorithmic proofs

$$A_n = \sum_{k=0}^n \underbrace{\binom{n}{k}^2 \binom{n+k}{k}^2}_{=: a_{n,k}} \text{ satisfies } (n+1)^3 A_{n+1} = (17n^2 + 17n + 5)(2n+1)A_n - n^3 A_{n-1}.$$

[van der Poorten, 1978]:

Neither Cohen nor I had been able to prove (S) or (S') in the intervening 2 months. After a few days of fruitless effort the specific problem was mentioned to Don Zagier (Bonn), and with irritating speed he showed that indeed the sequence  $\{b'_n\}$  satisfies the recurrence (4). This more or less broke the dam and (S) and (S') were quickly conquered.

# Algorithmic proofs

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> Zeilberger(a, n, k, N); finds in < 0.02 seconds:

$$L = (n+2)^3 N^2 - (17n^2 + 51n + 39)(2n+3)N + (n+1)^3 \text{ and}$$

$$C = (k^2 - 3/2k - 2n^2 - 6n - 4)k^4(16n+24)/(k-n-1)/(k-n-2),$$

with the property that  $(N \cdot a_{n,k} := a_{n+1,k}$  and  $K \cdot a_{n,k} := a_{n,k+1}$ ):

$$L \cdot \binom{n}{k}^2 \binom{n+k}{k}^2 = (K-1) \cdot C \binom{n}{k}^2 \binom{n+k}{k}^2.$$

Sum over  $k$  from 0 to  $n$  and conclude.



## Algorithmic proofs

$$m_n = \int_0^1 \underbrace{x^n \cdot \sqrt[3]{x(1-x)}}_{=: f_n(x)} dx \quad \text{satisfies} \quad \sum_{k \geq 0} m_k t^k = {}_2F_1 \left[ \begin{matrix} 1 & \frac{4}{3} \\ \frac{8}{3} \end{matrix}; t \right] \cdot \frac{2\pi^2}{15\Gamma(2/3)^3}.$$

> `creative_telescoping(f, n::shift, x::diff)`; finds in < 0.1 seconds:

$$L = (3n + 8)N - (3n + 4) \quad \text{and} \quad C(x) = 3x(x - 1),$$

with the property that  $(N \cdot f_n(x) = f_{n+1}(x))$ :

$$L \cdot x^n \sqrt[3]{x(1-x)} = \partial_x (C(x) \cdot x^n \sqrt[3]{x(1-x)})$$

It follows that  $L \cdot \int_0^1 x^n \sqrt[3]{x(1-x)} dx = 0$  and hence  $(3n + 8)m_{n+1} = (3n + 4)m_n$ .

# Algorithmic proofs

$$\int_0^{2\pi} \int_0^{2\pi} \frac{(\sqrt{2} + \sin v) du dv}{(1 + 2t(\sqrt{2} + \sin v) \cos u + t^2(3 + 2\sqrt{2} \sin v))^2}$$

$$= \frac{4\sqrt{2}\pi^2 (1 - t^2)}{(t^2 - 6t + 1)^2} {}_2F_1 \left[ \begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 \end{matrix}; \frac{4t}{(1 - t)^2} \right].$$

# Algorithmic proofs

$$\oint_{\gamma} \frac{2(2\sqrt{2}y - y^2 + 1)x \, dx \, dy}{(2\sqrt{2}t^2xy^2 + 2\sqrt{2}tx^2y - tx^2y^2 - 2\sqrt{2}t^2x - 2t^2xy + 2\sqrt{2}ty + tx^2 - ty^2 - 2yx + t)^2}$$

$$= \frac{4\sqrt{2}\pi^2(1-t^2)}{(t^2-6t+1)^2} {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 \end{matrix}; \frac{4t}{(1-t)^2}\right].$$

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$$= \frac{4\sqrt{2}\pi^2(1-t^2)}{(t^2-6t+1)^2} {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 \end{matrix}; \frac{4t}{(1-t)^2}\right].$$

> FindCreativeTelescoping[F, {Der[x], Der[y]}, Der[t]]; finds in 10 seconds:

$$L = t(3t^2 - 1)(9t^4 - 2t^2 + 1)(3t^2 + 1)^2 \partial_t^2 + (3t^2 + 1)(729t^8 + 162t^6 - 192t^4 + 38t^2 - 1) \partial_t + 12t(324t^8 + 333t^6 + 51t^4 - 53t^2 + 1), \text{ and}$$

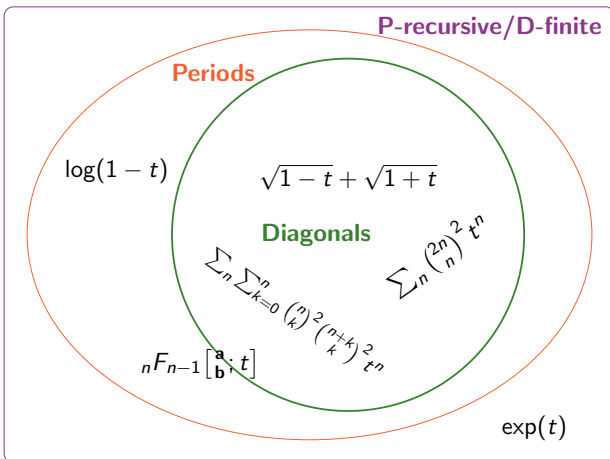
$C_1, C_2 \in \mathbb{Q}(x, y, t)$  with the property that:

$$L \cdot F = \partial_x C_1 + \partial_y C_2.$$

Therefore it follows that  $L \cdot \oint_{\gamma} F = 0$ . Solving  $Ly = 0$  we find the right-hand side.



## Definitions and interactions



A power series  $f(t) \in \mathbb{Q}[[t]]$  is **D-finite** if it satisfies a linear differential equation with polynomial coefficients:

$$p_n(t)f^{(n)}(t) + \cdots + p_0(t)f(t) = 0.$$

This equation can be rewritten:  $L \cdot f = 0$ ,

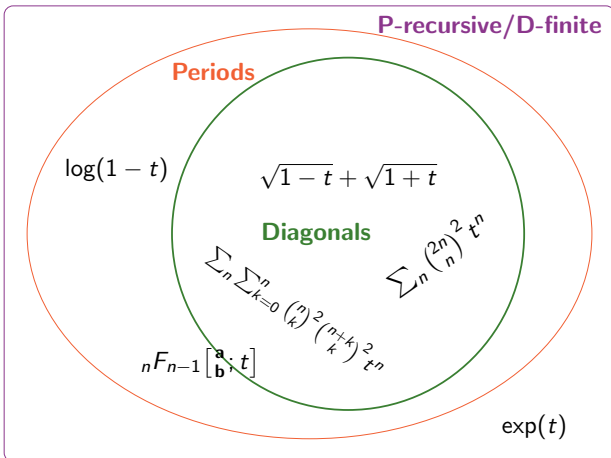
$$L = p_n(t)\partial_t^n + \cdots + p_0(t) \in \mathbb{Q}[t][[\partial_t]].$$

Let  $(\alpha)_n = \alpha \cdot (\alpha + 1) \cdots (\alpha + n - 1)$ .

Then  ${}_2F_1 \left[ \begin{matrix} a & b \\ c \end{matrix}; t \right] := \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n \cdot n!} t^n$  satisfies

$$t(1-t)f''(t) + (c - (a+b+1)t)f'(t) - abf(t) = 0.$$

# Definitions and interactions



A sequence  $(u_n)_{n \geq 0}$  is **P-recursive**, if it satisfies a linear recurrence with polynomial coefficients:

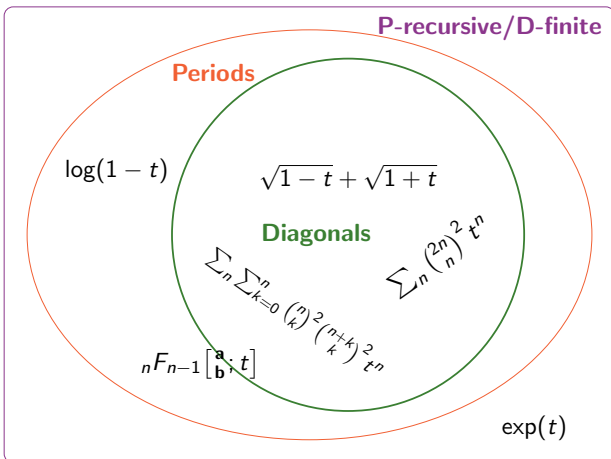
$$c_r(n)u_{n+r} + \cdots + c_0(n)u_n = 0.$$

Let  $(\alpha)_n = \alpha \cdot (\alpha + 1) \cdots (\alpha + n - 1)$ .

Then  $u_n = \frac{(a)_n \cdot (b)_n}{(c)_n \cdot n!}$  satisfies

$$(c+n)(n+1)u_{n+1} - (a+n)(b+n)u_n = 0.$$

## Definitions and interactions



A power series  $f(t) \in \mathbb{Q}[[t]]$  is called a **Period function** if it is an integral of a rational function in  $t$  and  $x_1, \dots, x_n$  over a semi-algebraic set.

$$p(t) = 4 \int_0^1 \sqrt{\frac{1-t^2x^2}{1-x^2}} dx$$

$$= 4 \iint \frac{dx dy}{1 - \frac{1-t^2x^2}{(1-x^2)y^2}} \text{ and}$$

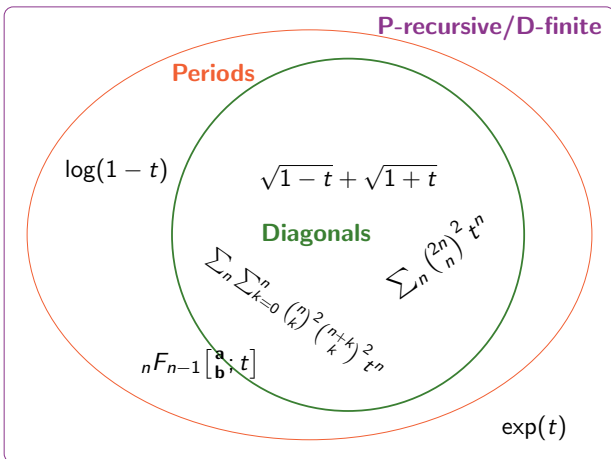
$$((t-t^3)\partial^2 + (1-t^2)\partial + t) \cdot p = 0,$$

$$p(t) = 2\pi - \frac{\pi}{2}t^2 - \frac{3\pi}{32}t^4 - \dots$$

**André-Bombieri-Katz's theorem:** A **Period function** is a G-function [André, 1989].

**Bombieri-Dwork conjecture:** Any G-function is a **Period function**.

## Definitions and interactions



A power series  $f(t) \in \mathbb{Q}[[t]] = \sum_k u_k t^k$  is called a **Diagonal** if there exists a rational function

$$R = \sum_{i_1, \dots, i_n \geq 0} c_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{Q}(x_1, \dots, x_n)$$

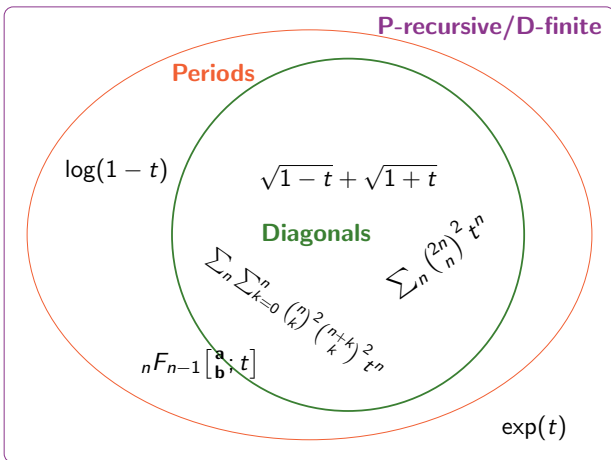
such that

$$f(t) = \text{Diag}(R) := \sum_{k \geq 0} c_{k, \dots, k} t^k.$$

Equivalently [Bostan, Lairez, Salvy 2017],  $(u_k)_{k \geq 0}$  is a **multiple binomial sum**.

$$\text{Diag} \frac{1}{1-x-y} = \text{Diag} \sum_{i,j \geq 0} \binom{i+j}{j} x^i y^j = \sum_{k \geq 0} \binom{2n}{n} t^k = \frac{1}{\sqrt{1-4t}}$$

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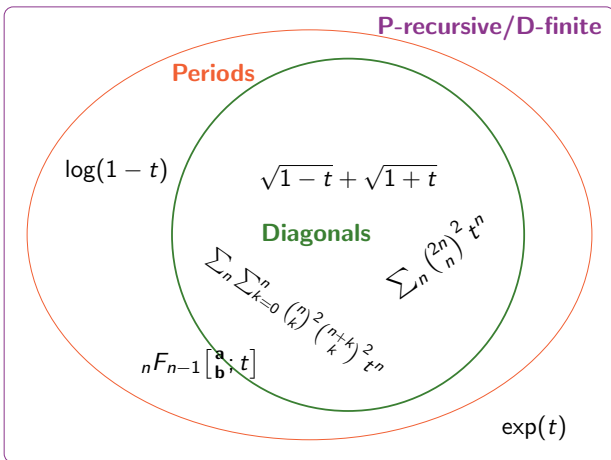
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$$\text{Diag} \frac{1}{1-x-y} = [x^{-1}] \frac{1}{x} \frac{1}{1-x-t/x} = \frac{1}{2\pi i} \oint_{|x|=\epsilon} \frac{dx}{x-x^2-t} = (1-4t)^{-\frac{1}{2}}$$

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**Christol's conjecture:** A convergent **D-finite** power series in  $\mathbb{Z}[[t]]$  is a **Diagonal**.

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- Goal: Given a **Period function** or **Diagonal**, find an annihilating ODE.

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$$L = p_n(t)\partial_t^n + \cdots + p_0(t) \in \mathbb{Q}[t][\partial_t], \quad \text{such that} \quad L \cdot \oint_{\gamma} R dx = 0.$$



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- Note:  $\int_{\gamma} \partial_{x_i} C dx = \int_{\partial\gamma} C dx = \int_{\emptyset} C dx = 0$  for any rational function  $C \in \mathbb{Q}(\mathbf{x}, t)$ .
- So we need to find

$$L \in \mathbb{Q}[t][\partial_t], \quad \text{and} \quad C_1, \dots, C_n \in \mathbb{Q}(x_1, \dots, x_n, t), \quad \text{such that}$$
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## Principle of Creative Telescoping

$$\sum_{k=0}^n p_k(t) \frac{d^k R}{dt^k} = \partial_{x_1} C_1 + \dots + \partial_{x_n} C_n \Rightarrow \left( \sum_{k=0}^n p_k(t) \partial_t^k \right) \cdot \oint_{\gamma} R dx = 0.$$

The **telescoper** and **certificates** always exist and can be found **algorithmically**.

# The Almkvist-Zeilberger algorithm [1990] *"I could never resist a definite integral."*

**Input:** A hyperexponential function  $H(t, x)$ , i.e.  $\partial_t H/H$  and  $\partial_x H/H \in \mathbb{Q}(t, x)$ .

**Output:** A linear differential operator  $P(t, \partial_t) \in \mathbb{Q}[t][\partial_t]$  and  $G(t, x) \in \mathbb{Q}(t, x)$ , s.t.

$$P \cdot H = \partial_x (G \cdot H).$$

**Algorithm:** Let  $\mathbb{L} = \mathbb{Q}(t)$ . For  $r = 0, 1, 2, \dots$  do:

- 1 Compute  $a(t, x) = \partial_x H/H$  and  $b_k(t, x) = \partial_t^k H/H$  for  $k = 0, \dots, r$ .
- 2 Decide whether the (ordinary, linear, inhomogeneous, parametrized) diff. equation

$$\partial_x G + a(t, x)G = \sum_{k=0}^r c_k(t)b_k(t, x)$$

has a rational solution  $G \in \mathbb{L}(x)$  for some  $c_0(t), \dots, c_r(t) \in \mathbb{L}$  not all zero.

- 3 If found solution in (2), return  $P = \sum_{k=0}^r c_k \partial_t^k$  and  $G$ ; else increase  $r$  and repeat.

# Some history of Creative Telescoping

- Indefinite integration/summation and working examples

- Sums: [Bernoulli, Fasenmyer, Gosper,...]

- Integrals: [Legendre, Ostrogradsky, Hermite, Picard, Manin, Griffiths, Feynman, ...]

$$\sum_{k=1}^n \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$$

$$\int \frac{P(x)}{Q(x)} dx = \frac{P_1(x)}{Q_1(x)} + \int \frac{P_2(x)}{Q_2(x)} dx$$

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)(1-xt)}} = \pi_2 F_1 \left[ \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix}; t \right]$$

- Algorithmic Creative Telescoping (algorithmic definite summation&integration):

- 1G: brutal elimination: [Fasenmyer, 1947], [Zeilberger, 1990], [Takayama, 1990]

- 2G: rational solutions of linear ODEs: [Zeilberger, 1990], [Almkvist,Zeilberger, 1990], [Chyzak, 2000], [Koutschan, 2010]

- 3G: 2G + linear algebra + bounds: [Apagodu, Zeilberger, 2005], [Koutschan 2010], [Chen, Kauers 2012], [Chen, Kauers, Koutschan 2014]

- 4G: based on (Hermite- and generalized Griffiths-Dwork) reduction [Bostan, Chen, Chyzak, Kauers, Koutschan, Li, Lairez, Salvy, Singer,...]

## Creative Telescoping and de Rham cohomology

*“the certificate is not needed, its existence and regularity are sufficient.”*

- Let  $\mathbb{L} = \mathbb{Q}(t)$ ,  $f \in L[x_0, \dots, x_n] = \mathbb{L}[x]$  and  $\gamma \subseteq \mathbb{C}^n$  a closed  $n$ -cycle.
- Denote by  $\mathbb{L}[x, 1/f]_p = \{F \in \mathbb{L}[x, 1/f] : F(\lambda x) = \lambda^p F(x), \forall \lambda \in \mathbb{Q}(t)\}$ .
- We wish to compute the differential equation satisfied by

$$\oint_{\gamma} F(t; x_0, \dots, x_n) dx, \text{ where } F = a/f^\ell \in \mathbb{L}[x, 1/f]_{-n-1}.$$

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- Therefore we wish to find a non-trivial element in

$$H_f^{\text{pr}} := \mathbb{L}[x, 1/f]_{-n-1} / D_f, \text{ where } D_f := \text{span}_{\mathbb{Q}}(\{\partial_{x_i} C : C \in \mathbb{L}[x, 1/f]_{-n}\})$$

- Generalized Griffiths-Dwork Reduction:  $F \mapsto [F]$ , s.t.  $\oint_{\gamma} F dx = 0 \iff [F] = 0$ .

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Theorem [Griffiths 1969, Bostan, Lairez, Salvy 2013, Lairez 2016]

Assume that  $\mathbb{L}[x] / \langle \partial_{x_0} f, \dots, \partial_{x_n} f \rangle$  is finite-dimensional over  $\mathbb{L}$ . Then  $H_f^{\text{PR}}$  is finitely generated over  $\mathbb{L}$ . Moreover the *Generalized Griffiths-Dwork Reduction* can be used to compute the (minimal regular) **telescoper**.

## Issues with singularities: non-regular certificates

*"the certificate is not needed, its existence and regularity are sufficient."*

- The following example originates in [Picard, 1899]: Let  $P_t(u) = u^3 + t$ , then

$$F = \frac{x - y}{z^2 - P_t(x)P_t(y)}$$

$$= \partial_x \frac{2P_t(x)}{(x - y)(z^2 - P_t(x)P_t(y))} + \partial_y \frac{2P_t(y)}{(x - y)(z^2 - P_t(x)P_t(y))} + \partial_z \frac{3(x^2 + y^2)z}{(x - y)(z^2 - P_t(x)P_t(y))},$$



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- So one has  $1 \cdot F = \partial_x C_1 + \partial_y C_2 + \partial_z C_3$ , **however:**

$$\oint_{\gamma} F \, dx \, dy \, dz \neq 0 \quad \text{for some } \gamma \subseteq \mathbb{C}^3.$$

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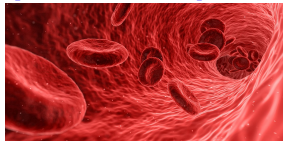
- Conclusion: Certificates are important.  
A certificate is called **regular** if it has no other poles than  $F$ .

# Motivation and Introduction

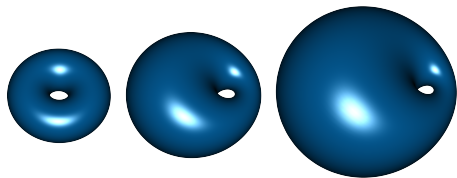
*“Why do all humans have the same biconcave shaped red blood cells?”*

- *Canham model* predicts shape of biomembranes like blood cells [Canham, 1970].
- The model asks to minimize the *Willmore energy*

$$W(S) := \int_S H^2 dA, \quad (H \text{ is the mean curvature})$$



over orientable closed surfaces  $S \subseteq \mathbb{R}^3$  with genus  $g$ , area  $A_0$  and volume  $V_0$ .

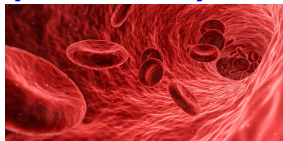


# Motivation and Introduction

*“Why do all humans have the same biconcave shaped red blood cells?”*

- *Canham model* predicts shape of biomembranes like blood cells [Canham, 1970].
- The model asks to minimize the *Willmore energy*

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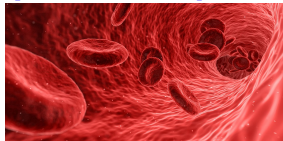
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Across all closed surfaces in  $\mathbb{R}^3$  of genus  $g \geq 1$  the Willmore energy is minimal for  $T_{\sqrt{2}}$ .

- $W(S)$  is invariant under Möbius transformations  $\Rightarrow$  no uniqueness of the shape.

## [Yu, Chen, 2021]: All projections of the (Clifford) torus

- The Clifford torus CT is defined as the following set in  $\mathbb{S}^3$ :

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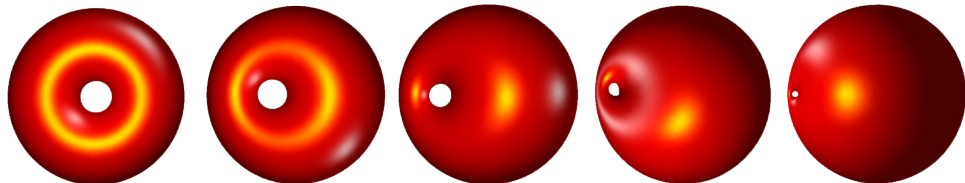
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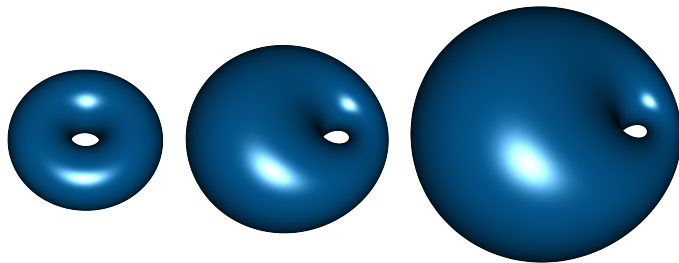
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- $\text{inv}_{(x,y,z)}$  is the inversion map about the unit sphere centered at  $(x, y, z) \in \mathbb{R}^3$ .
- The set of all shapes of stereographic projections of CT to  $\mathbb{R}^3$  is parameterized by

$$\{\text{inv}_{(t,0,0)}(T_{\sqrt{2}}) : t \in [0, \sqrt{2} - 1)\}.$$





$$W(\text{inv}_{(x,y,z)}(T)) = W(T) = \int_T H^2 dA = 2\pi^2.$$



## [Willmore, 1965] and [Marques, Neves, 2014]

Then we have

$$(17) \quad \tau(f) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} H^2 b(a + b \cos u) du dv.$$

After some computation we find, on writing  $b/a = c$ , that

$$(18) \quad \tau(f) = \frac{\pi}{2c\sqrt{1-c^2}}.$$

It is easy to see that  $\tau(f) \rightarrow \infty$  both as  $c \rightarrow 0$  and as  $c \rightarrow 1$ .

The minimum value of  $\tau(f)$  occurs when  $c = 1/\sqrt{2}$ , when the value of  $\tau(f)$  is  $\pi$ .

It seems reasonable to interpret  $\tau(f)$  as a measure of the „niceness“ of the shape of the surface  $f(S)$ , and to argue heuristically that a small value of  $\tau(f)$  corresponds to a simple shape for  $f(S)$ . This suggests that (13) with  $b/a = 1/\sqrt{2}$  gives the nicest shape for an embedded torus. However, whether or not  $\tau(T) = \pi$  remains an open question. The problem for surfaces of genus  $p \geq 2$  remains unsolved.

THE BLOG [SCIENCE](#)

## Math Finds the Best Doughnut

After a 47-year search, mathematicians Fernando C. Marques and André Neves have found the best doughnut, or at least the best geometric shape for a doughnut.

By Frank Morgan, Contributor

Atwell Professor of Mathematics, Emeritus, Williams College; Editor-in-Chief, Notices of the American Mathematical Society

[Marques, Neves, 2014]: Let  $\Sigma \subseteq \mathbb{S}^3$  be an embedded closed surface of genus  $g \geq 1$ . Then  $W(\Sigma) \geq 2\pi^2$  and the equality holds if and only if  $\Sigma$  is the Clifford torus up to conformal transformations of  $\mathbb{S}^3$ .

## Uniqueness with prescribed isoperimetric ratio

- In Canham's model, instead of  $A_0$  and  $V_0$  rather prescribe the *isoperimetric ratio*:

$$\iota_0 := \pi^{1/6} \frac{\sqrt[3]{6V_0}}{\sqrt{A_0}} \in (0, 1].$$

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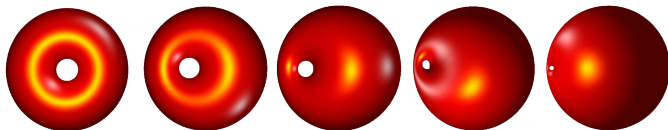
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Theorem (Yu, Chen, 22; Melczer, Mezzarobba, 22; Bostan, Y., 22)

*The shape of the projection of the Clifford torus to  $\mathbb{R}^3$  is uniquely determined by  $\iota_0$ . Thus, if  $g = 1$  and  $\iota_0^3 \in [3/(2^{5/4}\sqrt{\pi}), 1)$  then Canham's model has a unique solution.*



## Summary of [Yu, Chen, 22] and [Melczer, Mezzarobba, 22]

- Let  $\iota(S) := \pi^{1/6} \sqrt[3]{6V(S)/\sqrt{A(S)}} \in (0, 1]$ , and  $\tau := 3/(2^{5/4}\sqrt{\pi}) \approx 0.712$ . Define

$$\text{Iso}: [0, \sqrt{2} - 1) \rightarrow [\tau, 1),$$

$$t \mapsto \iota(\text{inv}_{(t,0,0)}(T_{\sqrt{2}}))^3$$

- $\sqrt{2}\pi^2 A(t^2)$  is the surface area and  $\sqrt{2}\pi^2 V(t^2)$  is the volume of  $\text{inv}_{(t,0,0)}(T_{\sqrt{2}})$ .

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- [Yu, Chen, 22]: Enough to show: Iso( $t$ ) is strictly increasing. Moreover,

$$\frac{V(t^2)A(t^2)}{2\pi^4} \frac{d}{dt} \ln(\text{Iso}(t)^2) = 72t + 1932t^3 + 31248t^5 + \dots =: \sum_{n \geq 0} a_n t^n$$

is a **D-finite** function. Enough to show:  $a_n > 0$  for all  $n \geq 0$ .

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- [Melczer, Mezzarobba, 22]: Rigorous asymptotics & error bounds:  $a_n > 0$ .

Therefore, Iso(t) is increasing.

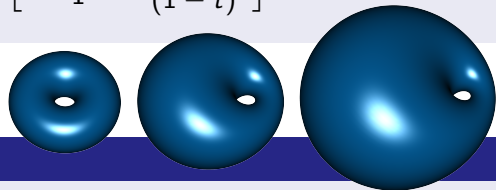
# Closed form solution

Proposition (Bostan, Y., 2022)

The surface area  $\sqrt{2}\pi^2 A(t^2)$  and volume  $\sqrt{2}\pi^2 V(t^2)$  of  $\text{inv}_{(t,0,0)}(T_{\sqrt{2}})$  are given by

$$A(t) = \frac{4(1-t^2)}{(t^2-6t+1)^2} \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 \end{matrix}; \frac{4t}{(1-t)^2}\right],$$

$$V(t) = \frac{2(1-t)^3}{(t^2-6t+1)^3} \cdot {}_2F_1\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} \\ 1 \end{matrix}; \frac{4t}{(1-t)^2}\right].$$



Corollary

The function  $\text{Iso}(t)^2 = 36\pi \frac{V(t^2)^2}{A(t^2)^3}$  is increasing on  $t \in (0, \sqrt{2}-1)$ .

# Proof of closed-form for $V(t)$

- Let  $Q(u, v, r; t) = 1 + 2t(\sqrt{2} + r \sin v) \cos u + t^2(2 + r^2 + 2\sqrt{2}r \sin v)t^2$ . Then

$$\begin{aligned}\sqrt{2}\pi^2 V(t^2) &= \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{r\sqrt{2} + r^2 \sin(v)}{Q(u, v, r; t)^3} du dv dr \\ &= \int_0^1 \oint_{|x|=|y|=1} F(x, y, r; t) dx dy dr = 2 + 48t^2 + \frac{1269}{2}t^4 + \dots\end{aligned}$$

for some  $F(x, y, r; t) \in \mathbb{Q}(x, y, r, t, \sqrt{2})$ . Thus  $V(t)$  is a **period function**.



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- First try:** Use creative telescoping on the triple integral:

```
> FindCreativeTelescoping[F, {Der[x], Der[y], Der[r]}, Der[t]];
```

finds  $C_1, C_2, C_3 \in \mathbb{Q}(x, y, r, t)$  such that  $F = \partial_x C_1 + \partial_y C_2 + \partial_r C_3$ .

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finds  $L \in \mathbb{Q}[r, t][\partial_t]$  and  $C_1, C_2 \in \mathbb{Q}(x, y, r, t)$  s.t.  $L \cdot F = \partial_x C_1 + \partial_y C_2$ .

- The common denominator of  $C_1$  and  $C_2$  is

$$\text{denom}(F) \cdot x \cdot y \cdot (1 + 2\sqrt{2}y - y^2) \cdot H(t, r).$$

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- The common denominator of  $C_1$  and  $C_2$  has

$$\text{denom}(F) \cdot x \cdot y \cdot (1 + 2\sqrt{2}y - y^2) \cdot H(t, r) \cap \gamma = \emptyset.$$

$$\sqrt{2}\pi^2 V(t^2) = \int_0^1 \underbrace{\oint_{|x|=|y|=1} F(x, y, r; t) dx dy}_{=: G(r, t)} dr.$$

$G(r, t)$  satisfies  $(P_2(r, t)\partial_r^2 + P_1(r, t)\partial_r + P_0(r, t))G(r, t) = 0$ . Then:

$$G(r, t) = Q_1 \cdot {}_2F_1\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} \\ 1 \end{matrix}; \phi_1\right] + Q_2 \cdot {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{3}{2} \\ 1 \end{matrix}; \phi_2\right],$$

for some (explicit)  $Q_1, Q_2, \phi_1, \phi_2 \in \mathbb{Q}(r, t)$ . Then we also find:

$$\int_0^s G(r, t) dr = \frac{2(s-t^2)^3}{(2-s)t^4 - 6t^2 + 1} \cdot {}_2F_1\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} \\ 1 \end{matrix}; \frac{4t^2 s}{(1-t^2(2-s))^2}\right].$$

Finally:  $\sqrt{2}\pi^2 V(t^2) = \int_0^1 G(r, t) dr$ , so set  $s = 1$  above.

# Iso is bijective

## Proposition

Let

$$A(t) = \frac{4(1-t^2)}{(t^2-6t+1)^2} \cdot {}_2F_1 \left[ \begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 \end{matrix}; \frac{4t}{(1-t)^2} \right],$$

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We need to show that

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is increasing on  $[0, 1)$ . Observe:  $h$  can be written as  $h(x) = \mathbf{g(x)^2 / f(x)^3}$ , where

$$\mathbf{g(x)} = {}_2F_1\left[\begin{matrix} -\frac{3}{2} & -\frac{3}{2} \\ 1 \end{matrix}; x\right] \cdot (x+1)^{-3/2} \quad \text{and} \quad \mathbf{f(x)} = {}_2F_1\left[\begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 \end{matrix}; x\right] \cdot (x+1)^{-1/2}$$

To show:  $\mathbf{g(x)}$  is increasing and  $\mathbf{f(x)}$  is decreasing on  $(0, 1)$ .



# Iso is bijective

## Proposition

Let  $a \geq 0$  and let  $w_a : [0, 1] \rightarrow \mathbb{R}$  be defined by

$$w_a(x) = {}_2F_1 \left[ \begin{matrix} -a & -a \\ 1 \end{matrix} ; x \right] \cdot (x+1)^{-a}.$$

Then  $w_a$  is: decreasing if  $0 < a < 1$ ; increasing if  $a > 1$ ; constant if  $a \in \{0, 1\}$ .

Clearly,  $\mathbf{g(x)} = w_{3/2}(x)$  and  $\mathbf{f(x)} = w_{-1/2}(x)$ .

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## Proof.

$$\frac{w'_a(x) \cdot (x+1)^{a+1}}{a \cdot (a-1) \cdot (1-x)^{2a}} = {}_2F_1 \left[ \begin{matrix} a+1 & a \\ 2 \end{matrix} ; x \right]. \quad \square$$

Clearly,  $\mathbf{g(x)} = w_{3/2}(x)$  and  $\mathbf{f(x)} = w_{-1/2}(x)$ .

# The general case $R > 1$

Recall:

$$T_R := \left\{ [(R + \cos v) \cos u, (R + \cos v) \sin u, \sin v]^T : u, v \in [0, 2\pi] \right\} \subseteq \mathbb{R}^3, \quad \text{and}$$

$\text{inv}_{(x,y,z)}$  is the inversion about the unit sphere centered at  $(x, y, z)$ .

## Question

Are there closed formulas for the volume and surface area of  $\text{inv}_{(x,y,z)}(T_R)$  for any  $R$ ?  
Is  $\text{Iso}_R(t)$  increasing in  $t$  for any  $R > 1$ ?

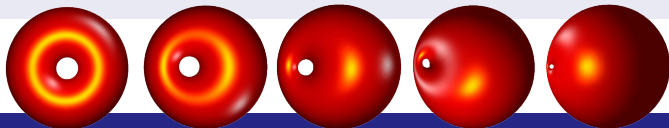
# Computing the isoperimetric ratio

Theorem (Bostan, Yu, Y., 2023)

The surface area  $A_R(t^2)R\pi^2$  and volume  $V_R(t^2)R\pi^2$  of  $\text{inv}_{(t,0,0)}\left(\frac{T_R}{R^2-1}\right)$  are given by

$$A_R(t) = \frac{4(1 - (R^2 - 1)t^2)}{(1 - 2(R^2 + 1)t + (R^2 - 1)^2 t^2)^2} \cdot {}_2F_1 \left[ \begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 \end{matrix}; \frac{4t}{(1 - (R^2 - 1)t)^2} \right],$$

$$V_R(t) = \frac{2(1 - (R^2 - 1)t^3)}{(1 - 2(R^2 + 1)t + (R^2 - 1)^2 t^2)^2} \cdot {}_3F_2 \left[ \begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2R^2-4} + 1 \\ 1 & \frac{3}{2R^2-4} \end{matrix}; \frac{4t}{(1 - (R^2 - 1)t)^2} \right].$$



Corollary

For  $R > 1$  the function  $\text{Iso}_R^2(t^2) = 36\pi \frac{V_R(t^2)^2}{A_R(t^2)^3}$  is increasing on  $t \in (0, (R + 1)^{-1})$ .

## Theorem

For  $R > 1$  the function  $\text{Iso}_R^2(t^2) = 36\pi \frac{V_R(t^2)^2}{A_R(t^2)^3}$  is increasing on  $t \in (0, (R+1)^{-1})$ , with

$$A_R(t) = \frac{4(1 - (R^2 - 1)t^2)}{(1 - 2(R^2 + 1)t + (R^2 - 1)^2 t^2)^2} \cdot {}_2F_1 \left[ \begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 \end{matrix}; \frac{4t}{(1 - (R^2 - 1)t^2)^2} \right],$$

$$V_R(t) = \frac{2(1 - (R^2 - 1)t^2)^3}{(1 - 2(R^2 + 1)t + (R^2 - 1)^2 t^2)^2} \cdot {}_3F_2 \left[ \begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2R^2 - 4} + 1 \\ 1 & \frac{3}{2R^2 - 4} \end{matrix}; \frac{4t}{(1 - (R^2 - 1)t^2)^2} \right].$$

First perform the substitution  $x = 4t^2 / ((1 - (R^2 - 1)t^2)^2)$ . It remains to show that:

$$h(x) := {}_3F_2 \left[ \begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2(R^2 - 2)} + 1 \\ 1 & \frac{3}{2(R^2 - 2)} \end{matrix}; x \right]^2 \cdot {}_2F_1 \left[ \begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 \end{matrix}; x \right]^{-3} \cdot (1 + (R^2 - 1) \cdot x)^{-3/2}$$

is increasing on  $x \in (0, 1)$  for all  $R > 1$ .

$$h(x) := {}_3F_2 \left[ \begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2(R^2-2)} + 1 \\ & 1 & \frac{3}{2(R^2-2)} \end{matrix} ; x \right]^2 \cdot {}_2F_1 \left[ \begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix} ; x \right]^{-3} \cdot (1 + (R^2 - 1) \cdot x)^{-3/2}$$

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is increasing on  $x \in (0, 1)$  for all  $R > 1$ . Note that  $h(x) = \mathbf{g(x)^2/f(x)^3}$ , where

$$g(x) := \frac{{}_3F_2 \left[ \begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2(R^2-2)} + 1 \\ & 1 & \frac{3}{2(R^2-2)} \end{matrix} ; x \right]}{(1+x)^{3/4} \cdot (1+(R^2-1) \cdot x)^{3/4}} \quad \text{and} \quad f(x) := {}_2F_1 \left[ \begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix} ; x \right] \cdot (x+1)^{-1/2}.$$

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We already saw:  $\mathbf{f(x)}$  is decreasing.



$$h(x) := {}_3F_2 \left[ \begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2(R^2-2)} + 1 \\ & 1 & \frac{3}{2(R^2-2)} \end{matrix} ; x \right]^2 \cdot {}_2F_1 \left[ \begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix} ; x \right]^{-3} \cdot (1 + (R^2 - 1) \cdot x)^{-3/2}$$

is increasing on  $x \in (0, 1)$  for all  $R > 1$ . Note that  $h(x) = \mathbf{g(x)^2/f(x)^3}$ , where

$$\mathbf{g(x)} := \frac{{}_3F_2 \left[ \begin{matrix} -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2(R^2-2)} + 1 \\ & 1 & \frac{3}{2(R^2-2)} \end{matrix} ; x \right]}{(1+x)^{3/4} \cdot (1 + (R^2 - 1) \cdot x)^{3/4}} \quad \text{and} \quad f(x) := {}_2F_1 \left[ \begin{matrix} -\frac{1}{2} & -\frac{1}{2} \\ & 1 \end{matrix} ; x \right] \cdot (x+1)^{-1/2}.$$

We already saw:  $\mathbf{f(x)}$  is decreasing. For  $\mathbf{g(x)}$  it holds that:

$$\frac{4 \cdot \mathbf{g'(x)} \cdot (1+x)^{7/4} \cdot (1 + (R^2 - 1) \cdot x)^{7/4}}{3 \cdot (1-x)^2 \cdot (R^2 - 1)} =: \sum_{n \geq 0} u_n(R) x^n, \quad \text{and}$$

$u_{n+1}(R)/u_n(R) = (2n-1)(2n+1) p_{n+1}(R)/(4(n+2)(n+1) p_n(R))$ ,  $u_0(R) = 1$ , where

$$p_n(R) := 4(R^4 + 4R^2 - 4)n^3 + 6(R^4 + R^2 - 2)n^2 + (2R^4 - 13R^2 + 10)n - 3R^2 + 3 > 0.$$

## Summary and conclusion

- Creative Telescoping is a powerful tool for dealing with **Period functions**.
- Implemented versions of Creative Telescoping exist (both 2G and 4G). They are useful in practice and can solve non-trivial problems.
- The surface area and volume of any stereographic projection to  $\mathbb{R}^3$  of the Clifford torus can be expressed in terms of **hypergeometric functions**.
- The Canham model in genus 1 has a unique solution when  $\iota_0^3 \in \left(\frac{3}{2^{5/4}}\pi^{-\frac{1}{2}}, 1\right)$ .