On a class of hypergeometric diagonals[1] [2]

Sergey Yurkevich

University of Vienna

Friday 25th September, 2020

[1] Joint work with Alin Bostan, arxiv.org/2008.12809

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Bonjour, groupe de travail « Transcendance et combinatoire » !

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Overview

On a class of hypergeometric **diagonals**

Given a multivariate power series

$$g(x_1, \dots, x_n) = \sum_{(i_1,\dots,i_n)\in\mathbb{N}^n} g_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n} \in \mathbb{Q}[\![x_1, \dots, x_n]\!],$$

define the *diagonal* $\mathrm{Diag}(g)$ as the univariate power series given by

$$\mathrm{Diag}(g) := \sum_{j\geq 0} g_{j,\dots,j} t^j.$$

Diagonals ($n = 2$)

$$g(x,y) = \begin{matrix} \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ g_{0,4}x^0y^4 & g_{1,4}x^1y^4 & g_{2,4}x^2y^4 & g_{3,4}x^3y^4 & g_{4,4}x^4y^4 & \cdots \\ g_{0,3}x^0y^3 & g_{1,3}x^1y^3 & g_{2,3}x^2y^3 & g_{3,3}x^3y^3 & g_{4,3}x^4y^3 & \cdots \\ g_{0,2}x^0y^2 & g_{1,2}x^1y^2 & g_{2,2}x^2y^2 & g_{3,2}x^3y^2 & g_{4,2}x^4y^2 & \cdots \\ g_{0,1}x^0y^1 & g_{1,1}x^1y^1 & g_{2,1}x^2y^1 & g_{3,1}x^3y^1 & g_{4,1}x^4y^1 & \cdots \\ g_{0,0}x^0y^0 & g_{1,0}x^1y^0 & g_{2,0}x^2y^0 & g_{3,0}x^3y^0 & g_{4,0}x^4y^0 & \cdots \end{matrix}$$

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$$\mathrm{Diag}(g(x,y)) = g_{0,0} + g_{1,1}t + g_{2,2}t^2 + g_{3,3}t^3 + g_{4,4}t^4 + \cdots .$$

Examples

- Let $g(x, y) = 1/(1 - x - y)$. Then

$$\mathrm{Diag}(g) = \mathrm{Diag}\left(\sum_{i,j\geq 0}\binom{i+j}{i}x^i y^j\right) = \sum_{n\geq 0}\binom{2n}{n}t^n = (1-4t)^{-1/2}.$$

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- The Apéry numbers [Straub, 2014]:

$$\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 t^n = \mathrm{Diag}\left(\frac{1}{(1-x_1-x_2)(1-x_3-x_4)-x_1x_2x_3x_4}\right).$$

Hadamard product

The Hadamard product of two univariate power series:

$$(f_0 + f_1\, t + f_2\, t^2 + \cdots) \star (h_0 + h_1\, t + h_2\, t^2 + \cdots) = f_0 h_0 + f_1 h_1\, t + f_2 h_2\, t^2 + \cdots.$$

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Proposition

For all multivariate power series $g(x_1, \ldots, x_n)$ *and* $h(y_1, \ldots, y_m)$ *we have*

$$\mathrm{Diag}(g(x_1, \ldots, x_n) \cdot h(y_1, \ldots, y_m)) = \mathrm{Diag}(g(x_1, \ldots, x_n)) \star \mathrm{Diag}(h(y_1, \ldots, y_m)).$$

Definitions

- $g(x_1, \ldots x_n) \in \mathbb{Q}[\![x_1, \ldots, x_n]\!]$ is rational if $g = P(x_1, \ldots, x_n)/Q(x_1, \ldots, x_n)$ for polynomials P, Q.
- $g(x_1, \ldots x_n)$ is algebraic if there exists a non-zero polynomial $P(x_1, \ldots, x_n, t)$ such that $P(x_1, \ldots, x_n, g) = 0$.
- $f(t)$ is D-finite (holonomic) if f is the solution of a linear ODE with polynomial coefficients.
- $f(t) \in \mathbb{Q}[\![t]\!]$ is globally bounded if f has non-zero radius of convergence and there exists $\alpha, \beta \in \mathbb{N}$ such that $\alpha f(\beta t) \in \mathbb{Z}[\![t]\!]$.

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- $\mathrm{DIAG}_r = \{f(t) \in \mathbb{Q}[\![t]\!] : \exists$ rational $g(x_1, \dots, x_n)$ such that $f = \mathrm{Diag}(g)\}$
- $\mathrm{DIAG}_a = \{f(t) \in \mathbb{Q}[\![t]\!] : \exists$ algebraic $g(x_1, \dots, x_n)$ such that $f = \mathrm{Diag}(g)\}$

Properties, theorems and facts

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- If $g(x_1, \ldots, x_n)$ is rational, then the coefficient sequence of $f(t) = \mathrm{Diag}(g)$ is a *multiple binomial sum*. The converse is also true. [Bostan, Lairez, Salvy, 2016]

Open questions about diagonals

- Describe the set DIAG, i.e. which series $f(t)$ can be written as diagonals of rational/algebraic multivariate functions $g(x_1, \ldots, x_n)$?
- How many variables do we need at least to represent $f(t)$ as the diagonal of some algebraic/rational $g(x_1, \ldots, x_n)$?

Christol's conjecture

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(C) Conjecture [Christol, 1987]: If a power series $f \in \mathbb{Q}[\![t]\!]$ is D-finite and globally bounded then $f \in \mathrm{DIAG}$, i.e. $f = \mathrm{Diag}(g)$ for some rational power series $g \in \mathbb{Q}[\![x_1, \ldots, x_n]\!]$.

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- If $f(t) = f_1(t) \star f_2(t)$ for algebraic series $f_1(t)$ and $f_2(t)$, then f is both D-finite and globally bounded. Christol's conjecture holds again:

$$f(t) = f_1 \star f_2 = \mathrm{Diag}(g_1(x_1, x_2)) \star \mathrm{Diag}(g_2(y_1, y_2)) = \mathrm{Diag}(g_1(x_1, x_2) \cdot g_2(y_1, y_2)).$$

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- The *Hadamard grade* [Allouche and Mendès-France, 2011] of a power series $f(t)$ is the least positive integer $h = h(f)$ such that $f(t)$ can be written as the Hadamard product of h algebraic power series.

On a class of **hypergeometric** diagonals

Let $(x)_j := x(x+1)\cdots(x+j-1)$ be the rising factorial.
The *(generalized) hypergeometric function* ${}_pF_q$ with rational parameters $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ is the univariate power series in $\mathbb{Q}[\![t]\!]$ defined by

$$ {}_pF_q([a_1, \ldots, a_p], [b_1, \ldots, b_q]; t) := \sum_{j \geq 0} \frac{(a_1)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \frac{t^j}{j!}. $$

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The *height* of such a hypergeometric function is given by

$$ h = |\{1\leqslant j\leqslant q+1 \mid b_j\in\mathbb{Z}\}| - |\{1\leqslant j\leqslant p \mid a_j\in\mathbb{Z}\}|, $$

where $b_{q+1}=1$.

Examples

- For all $a \in \mathbb{Q}$ we have ${}_1F_0([a];[\,];t) = 1 + \frac{a}{1}t + \frac{a\cdot(a+1)}{1\cdot 2}t^2 + \cdots = (1-t)^{-a}$.

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- $$\begin{aligned}{}_2F_1\left(\left[\frac{1}{3},-\frac{1}{6}\right];\left[\frac{3}{2}\right],t\right) &= 1 + \frac{(1/3)\cdot(-1/6)}{(3/2)\cdot 1}t + \frac{(1/3)(4/3)\cdot(-1/6)(5/6)}{(3/2)(5/2)\cdot 1\cdot 2}t^2 + \cdots \\ &= \frac{(1+\sqrt{t})^{1/3} + (1-\sqrt{t})^{1/3}}{2}.\end{aligned}$$

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- $$ {}_2F_1\left(\left[\frac{1}{3}, -\frac{1}{6}\right]; \left[\frac{3}{2}\right], t\right) = 1 + \frac{(1/3)\cdot(-1/6)}{(3/2)\cdot 1}t + \frac{(1/3)(4/3)\cdot(-1/6)(5/6)}{(3/2)(5/2)\cdot 1\cdot 2}t^2 + \cdots $$
  $$ = \frac{(1+\sqrt{t})^{1/3} + (1-\sqrt{t})^{1/3}}{2}. $$
- ${}_3F_2([1,1,1];[2,2];t) = \mathrm{Li}_2(t)/t = \sum_{n\geq 1} \frac{t^{n-1}}{n^2}$.

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Properties of hypergeometric functions

- ${}_{p+1}F_{q+1}([a_1, \ldots, a_p, c], [b_1, \ldots, b_q, c]; t) = {}_pF_q([a_1, \ldots, a_p], [b_1, \ldots, b_q]; t).$

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&{}_{p_1}F_{q_1}([a_1, \dots, a_{p_1}], [b_1, \dots, b_{q_1}]; t) \star {}_{p_2}F_{q_2}([c_1, \dots, c_{p_2}], [d_1, \dots, d_{q_2}]; t) \\
&= {}_{p_1+p_2}F_{q_1+q_2+1}([a_1, \dots, a_{p_1}, c_1, \dots, c_{p_2}], [b_1, \dots, b_{q_1}, d_1, \dots, d_{q_2}, 1]; t).
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- ${}_pF_q$ is not a polynomial and globally bounded $\Rightarrow q = p - 1$.

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\end{aligned}
$$

- All hypergeometric functions are D-finite.
- ${}_pF_q$ is not a polynomial and globally bounded $\Rightarrow q = p - 1$.
- The case when ${}_pF_q([a_1, \dots, a_p], [b_1, \dots, b_q]; t)$ is algebraic is completely classified [Schwarz, 1873; Beukers and Heckman, 1989]

Algebraicity of the hypergeometric function

Theorem (Interlacing criterion: Beukers and Heckman, 1989)

Assume that the rational parameters $\{a_1, \dots, a_p\}$ *and* $\{b_1, \dots, b_{p-1}, b_p = 1\}$ *are disjoint modulo* $\mathbb{Z}$. *Let N be their common denominator. Then*

$${}_pF_{p-1}([a_1, \dots, a_p], [b_1, \dots, b_{p-1}]; t)$$

is algebraic if and only if for all $1 \le r < N$ *with* $\gcd(r, N) = 1$ *the numbers* $\{\exp(2\pi i r a_j), 1 \le j \le p\}$ *and* $\{\exp(2\pi i r b_j), 1 \le j \le p\}$ *interlace on the unit circle*

Interlacing criterion in practice I

- Take $f(t) = {}_3F_2([1/4, 3/8, 7/8], [1/3, 2/3]; t)$. Is $f(t)$ algebraic?

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- Take $f(t) = {}_3F_2([1/4, 3/8, 7/8], [1/3, 2/3]; t)$. Is $f(t)$ algebraic?
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- We have $\varphi(24) = 8$, and each $r \in \{1, 5, 7, 11, 13, 17, 19, 23\} =: S$ is coprime to 24.

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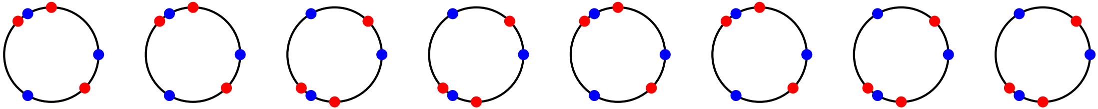

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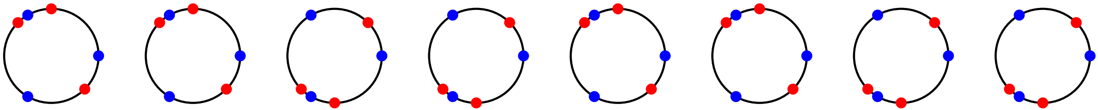

$\Rightarrow f(t)$ is algebraic.

Global boundedness of the hypergeometric function

Theorem (Christol, 1986)

Assume that the rational parameters $\{a_1, \ldots, a_p\}$ *and* $\{b_1, \ldots, b_{p-1}, b_p = 1\}$ *are disjoint modulo* $\mathbb{Z}$. *Let N be their common denominator. Then*

$$ {}_pF_{p-1}([a_1, \ldots, a_p], [b_1, \ldots, b_{p-1}]; t) $$

is globally bounded if and only if for all $1 \leq r < N$ *with* $\gcd(r, N) = 1$, *one encounters more numbers in* $\{\exp(2\pi i r a_j), 1 \leq j \leq p\}$ *than in* $\{\exp(2\pi i r b_j), 1 \leq j \leq p\}$ *when running through the unit circle from* 1 *to* $\exp(2\pi i)$.

Interlacing criterion in practice II

- Is $f(t) = {}_3F_2([1/9, 4/9, 5/9], [1/3, 1/2]; t)$ algebraic or at least globally bounded?

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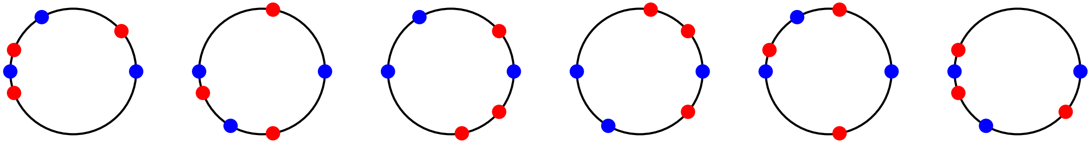

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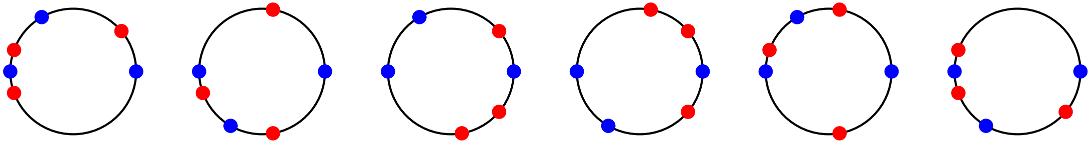
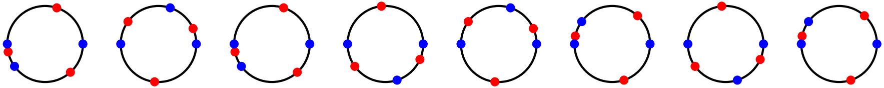


$\Rightarrow f(t)$ is transcendental and not even globally bounded.

Commercial break

Commercial break

TikZ: `\BeHe{30}{1/5}{8/15}{13/15}{1/2}{3/5}`



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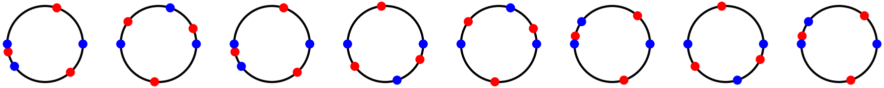

Maple: ishyperdiagalgebraic([[1/5, 8/15, 13/15], [1/2, 3/5]])

>true

Christol's conjecture and hypergeometric functions

(C) If a power series $f \in \mathbb{Q}[\![t]\!]$ is D-finite and globally bounded then $f = \mathrm{Diag}(g)$ for some algebraic power series $g \in \mathbb{Q}[\![x_1, \ldots, x_n]\!]$.

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- (C') If a hypergeometric function ${}_pF_{p-1}([a_1, \ldots, a_p], [b_1, \ldots, b_{p-1}]; t) \in \mathbb{Q}[\![t]\!]$ is globally bounded then $f = \mathrm{Diag}(g)$ for some algebraic power series $g \in \mathbb{Q}[\![x_1, \ldots, x_n]\!]$.

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(C''') Show that

$$ {}_3F_2\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\right], \left[\frac{1}{3}, 1\right]; 729\, t\right) = 1 + 60\, t + 20475\, t^2 + 9373650\, t^3 + \cdots $$

is the diagonal of some algebraic $g \in \mathbb{Q}[\![x_1, \ldots, x_n]\!]$.

Hypergeometric function and Christol's conjecture: resolved cases

Recall that the height of $f(t) = {}_pF_{p-1}([a_1, \ldots, a_p], [b_1, \ldots, b_{p-1}]; t)$ is given by

$$h = |\{1 \leqslant j \leqslant p \mid b_j \in \mathbb{Z}\}| - |\{1 \leqslant j \leqslant p \mid a_j \in \mathbb{Z}\}|,$$

where $b_p = 1$.

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where $b_p = 1$.

- Assume that $h = 1$ (all b_i's are non-integer). Then [Beukers and Heckman, 1989; Christol, 1990]

$$f \text{ algebraic} \iff f \text{ globally bounded.}$$

- Assume that $h = p$ (all b_i's are integer). Then

$$f(t) = {}_1F_0([a_1], [\,], t) \star {}_1F_0([a_2], [\,], t) \star \cdots \star {}_1F_0([a_p], [\,], t).$$

Each ${}_1F_0([a_j], [\,], t) = (1 - t)^{-a_j}$ is algebraic.

First non-trivial example: ${}_3F_2([a, b, c], [d, 1]; t)$

- Assume $f(t) = {}_3F_2([a, b, c], [d, 1]; t)$ is globally bounded. Is $f(t)$ a diagonal?

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- It always holds that

$$f(t) = {}_2F_1([a, b], [d]; t) \star {}_1F_0([c], [\,]; t) = {}_2F_1([a, c], [d]; t) \star {}_1F_0([b], [\,]; t)$$
$$= {}_2F_1([b, c], [d]; t) \star {}_1F_0([a], [\,]; t),$$

therefore we can assume that each such ${}_2F_1$ is transcendental.

First non-trivial example: ${}_3F_2([a,b,c],[d,1];t)$

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$$= {}_2F_1([b,c],[d];t) \star {}_1F_0([a],[\,];t),$$

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(C^{iv}) List with 116 such ${}_3F_2$'s by [Bostan, Boukraa, Christol, Hassani, Maillard, 2011]:

$$\mathrm{BBCHM} = \{{}_3F_2([1/3,5/9,8/9],[1/2,1];t), {}_3F_2([1/4,3/8,5/6],[2/3,1];t), \ldots,$$
$$\ldots, {}_3F_2([1/9,4/9,5/9],[1/3,1];t), \ldots\}$$

A class of hypergeometric diagonals

- Main question: when can we write $f(t) \in \mathbb{Q}[\![t]\!]$ as the diagonal of some rational/algebraic $g(x_1, \ldots, x_n) \in \mathbb{Q}[\![x_1, \ldots, x_n]\!]$?
- First non-trivial/unsolved class:

$$f(t) = {}_3F_2([a, b, c], [d, 1]; t),$$

  such that $f(t)$ is globally bounded.
- Explicit example [Christol, 1986]:

$$f(t) = {}_3F_2([1/9, 4/9, 5/9], [1/3, 1]; t).$$

- List of 116 similar "difficult" examples [BBCHM, 2011].

A class of hypergeometric diagonals

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- List of 116 similar “difficult” examples [BBCHM, 2011].
- Recent progress by [Abdelaziz, Koutschan, Maillard, 2020]:

$${}_3F_2([1/9, 4/9, 7/9], [1/3, 1]; t) \quad \text{and} \quad {}_3F_2([2/9, 5/9, 8/9], [2/3, 1]; t)$$

are diagonals.

Result of Abdelaziz, Koutschan and Maillard, 2020

$$ {}_3F_2\left(\left[\frac{1}{9},\frac{4}{9},\frac{7}{9}\right],\left[\frac{1}{3},1\right];27\,t\right)=\mathrm{Diag}\left(\frac{(1-x-y)^{2/3}}{1-x-y-z}\right), \qquad \text{and} $$

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$$

More generally,

$$
{}_3F_2\left(\left[\frac{1-R}{3},\frac{2-R}{3},\frac{3-R}{3}\right],[1,1-R];27\,t\right) = \operatorname{Diag}\left(\frac{(1-x-y)^{R}}{1-x-y-z}\right),
$$

for all $R \in \mathbb{Q}$.

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More generally,

$$ {}_3F_2\left(\left[\frac{1-R}{3},\frac{2-R}{3},\frac{3-R}{3}\right],[1,1-R];-27\,t\right) = \mathrm{Diag}\left((1+x+y)^R(1+x+y+z)^{-1}\right), $$

for all $R \in \mathbb{Q}$.

Main result I

Theorem (Bostan and Y., 2020)

Let $N \in \mathbb{N} \setminus \{0\}$ *and* $b_1, \ldots, b_N \in \mathbb{Q}$ *with* $b_N \neq 0$. *Then*

$$\mathrm{Diag}((1+x_1)^{b_1}(1+x_1+x_2)^{b_2}\cdots(1+x_1+\cdots+x_N)^{b_N})$$

is a hypergeometric function.

Complete identity

Let $B(k) := -(b_k + \cdots + b_N)$.

$$u^k := \left(\frac{B(k)}{N-k+1}, \frac{B(k)+1}{N-k+1}, \ldots, \frac{B(k)+N-k}{N-k+1} \right), \quad k = 1, \ldots, N,$$

$$v^k := \left(\frac{B(k)}{N-k}, \frac{B(k)+1}{N-k}, \ldots, \frac{B(k)+N-k-1}{N-k} \right), \quad k = 1, \ldots, N-1.$$

Set $v^N := (1, 1, \ldots, 1)$ with $N-1$ ones and $M := N(N+1)/2$. Define $u := [u^1, \ldots, u^N]$ and $v := [v^1, \ldots, v^N]$.

Theorem (Bostan and Y., 2020)

It holds that

$$\mathrm{Diag}((1+x_1)^{b_1}(1+x_1+x_2)^{b_2} \cdots (1+x_1+\cdots+x_N)^{b_N}) = {}_M F_{M-1}(u; v; (-N)^N t).$$

Examples

- If $N = 2$ we have

$$\mathrm{Diag}\left((1+x)^R(1+x+y)^S\right) = {}_3F_2\left(\left[\frac{-(R+S)}{2}, \frac{-(R+S)+1}{2}, -S\right]; [-(R+S), 1]; 4t\right).$$

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- And

$$\mathrm{Diag}\left((1+x)^{1/4}(1+x+y)^{-3/4}\right) = {}_3F_2\left(\left[\frac{1}{4}, \frac{3}{4}, \frac{3}{4}\right]; \left[\frac{1}{2}, 1\right]; 4t\right).$$

Examples

- Letting $N = 3$ we obtain

$$\mathrm{Diag}\left((1+x)^R(1+x+y)^S(1+x+y+z)^T\right) =$$

$${}_6F_5\left(\left[\frac{-(R+S+T)}{3}, \frac{-(R+S+T)+1}{3}, \frac{-(R+S+T)+2}{3}, \frac{-(S+T)}{2}, \frac{-(S+T)+1}{2}, -T\right];\right.$$
$$\left.\left[\frac{-(R+S+T)}{2}, \frac{-(R+S+T)+1}{2}, -(S+T), 1, 1\right]; -27t\right).$$

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- Hence

$$\mathrm{Diag}\left(\frac{(1+x+y)^S}{1+x+y+z}\right) = {}_3F_2\left(\left[\frac{1-S}{3}, \frac{2-S}{3}, \frac{3-S}{3}\right], [1, 1-S]; -27\,t\right).$$

Main lemma

Lemma

Let N be a positive integer and $b_1, \ldots, b_N \in \mathbb{Q}$ such that $b_N \neq 0$. It holds that

$$[x_1^{k_1} \cdots x_N^{k_N}](1+x_1)^{b_1} \cdots (1+x_1+\cdots+x_N)^{b_N} = \binom{b_N}{k_N}\binom{b_{N-1}+b_N-k_N}{k_{N-1}} \cdots \binom{b_1+\cdots+b_N-k_N-\cdots-k_2}{k_1}.$$

Proof of main lemma

Proof.

$$[x_1^{k_1} \cdots x_{N-1}^{k_{N-1}} \cdot x_N^{k_N}](1+x_1)^{b_1} \cdots (1+x_1+\cdots+x_{N-1})^{b_{N-1}}(1+x_1+\cdots+x_N)^{b_N}$$
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□

Sketch of proof of Main Theorem

- To show:

$$\mathrm{Diag}((1+x_1)^{b_1}(1+x_1+x_2)^{b_2}\cdots(1+x_1+\cdots+x_N)^{b_N}) = {}_MF_{M-1}(u;v;(-N)^N t).$$

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$$[t^n]{}_MF_{M-1}(u;v;(-N)^N t) = (-1)^{Nn}N^{Nn}\frac{\prod_{i,j}(u_j^{(i)})_n}{\prod_{i,j}(v_j^{(i)})_n \cdot n!}$$

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Algebraicity of $\mathrm{Diag}((1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_N)^{b_N})$

Corollary

Let $f(t) = \mathrm{Diag}((1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_N)^{b_N})$, *then* f *is algebraic if and only if* $N = 2$ *and* $b_2 \in \mathbb{Z}$, *or* $N = 1$.

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Sketch of proof.

$$ {}_MF_{M-1}(u;v;t) = {}_MF_{M-1}([u^{(1)},\dots,u^{(N)}];[v^{(1)},\dots,v^{(N-1)},\underbrace{1,1,\dots,1}_{N-1 \text{ times}}];t). $$

We can have at most one cancellation between $u^{(k)}$ and a 1. By Christol's theorem, $N \leq 2$. □

Hadamard grade of $\operatorname{Diag}((1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_N)^{b_N})$

Recall that the *Hadamard grade* of $f(t)$ is the least positive integer $h = h(f)$ such that $f(t)$ can be written as the Hadamard product of h algebraic power series.

Corollary

The Hadamard grade of $\operatorname{Diag}((1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_N)^{b_N})$ *is finite and* $\leq N$.

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Proof.

$${}_MF_{M-1}(u;v;t) = {}_NF_{N-1}(u^{(1)};v^{(1)};t) \star {}_{N-1}F_{N-2}(u^{(2)};v^{(2)};t) \star \cdots \star {}_1F_0(u^{(N)};;t), \quad \text{and}$$

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$${}_{N-k+1}F_{N-k}(u^{(k)};v^{(k)};t) = {}_NF_{N-1}\left(\left[\frac{B(k)}{N-k+1},\frac{B(k)+1}{N-k+1},\ldots,\frac{B(k)+N-k}{N-k+1}\right];\right.$$
$$\left.\left[\frac{B(k)}{N-k},\frac{B(k)+1}{N-k},\ldots,\frac{B(k)+N-k-1}{N-k}\right];t\right)$$

is algebraic. □

The list BBCHM

(C^{iv}) Bostan, Boukraa, Christol, Hassani and Maillard produced in 2011 a list with 116 ${}_3F_2$'s such that:

- ${}_3F_2([a,b,c],[d,1];t)$ is globally bounded.
- $a,b,c,d \in \mathbb{Q} \setminus \mathbb{Z}$, distinct mod $\mathbb{Z}$, and $0 < a,b,c,d < 1$.
- Each ${}_2F_1([a,b],[d];t), {}_2F_1([a,c],[d];t), {}_2F_1([b,c],[d];t)$ is transcendental.

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for *any* $r \in \mathbb{Q}$.
- If for *some* r, both ${}_2F_1([a,b],[r];t)$ and ${}_2F_1([c,r],[d];t)$ are algebraic, or ${}_3F_2([a,b,c],[d,r];t)$ is algebraic, then f is a diagonal.

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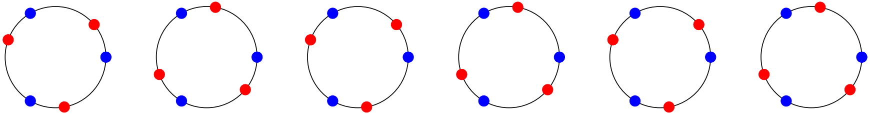

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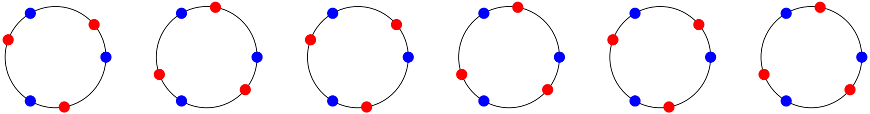

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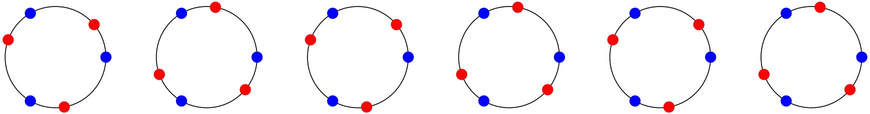

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New Results on the BBCHM list

- By writing

$$ {}_3F_2([a,b,c],[d,1];t) = {}_2F_1([a,b],[r];t) \star {}_2F_1([c,r],[d];t) = \dots, $$

and searching for $r \in \mathbb{Q}$ such that ${}_2F_1([a,b],[r];t)$ and ${}_2F_1([c,r],[d];t)$ are algebraic, we can resolve 28 cases of the list. Then $116 - 28 = 88$ remain.

- By writing

$$ {}_3F_2([a,b,c],[d,1];t) = {}_3F_2([a,b,c],[d,r];t) \star {}_1F_0([r],[\,];t), $$

we can resolve 12 more cases. So $88 - 12 = 76$ remain.

Limitations

- We can also write

$$
\begin{aligned}
{}_3F_2([a,b,c],[d,1];t) &= {}_3F_2([a,b,s],[d,r];t) \star {}_2F_1([c,r],[s];t) = \dots \\
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$$ = {}_3F_2([a,b,c],[r,s];t) \star {}_2F_1([r,s],[d];t), $$

however this does not resolve any new cases.

- Assuming the Rohrlich-Lang conjecture, [Rivoal and Roques, 2014] could prove that

$$ {}_3F_2\left(\left[\frac{1}{7},\frac{2}{7},\frac{4}{7}\right],\left[1,\frac{1}{2}\right],2401\,t\right) = 1 + 112\,t + 103488\,t^2 + 139087872\,t^3 + \cdots $$

has infinite Hadamard grade.

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- 40 cases of the list BBCHM are resolved.
- Considerations with the Hadamard grade show that we need a new viewpoint.

Main result II

Theorem (Bostan and Y., 2020)

Let $N \in \mathbb{N} \setminus \{0\}$ and $b_1, \ldots, b_N \in \mathbb{Q}$ with $b_N \neq 0$ and $b_{N-1} + b_N = -1$. Then for any $b \in \mathbb{Q}$,

$$\mathrm{Diag}((1+x_1)^{b_1}(1+x_1+x_2)^{b_2} \cdots (1+x_1+\cdots+x_N)^{b_N} \cdot (1+x_1+\cdots+2\,x_{N-1})^{b})$$

is a hypergeometric function.

Complete identity

Let $B(k) := -(b_k + \cdots + b_N + b)$.

$$u^k := \left(\frac{B(k)}{N-k+1}, \frac{B(k)+1}{N-k+1}, \ldots, \frac{B(k)+N-k}{N-k+1}\right), \quad k = 1, \ldots, N-2$$

$$v^k := \left(\frac{B(k)}{N-k}, \frac{B(k)+1}{N-k}, \ldots, \frac{B(k)+N-k-1}{N-k}\right), \quad k = 1, \ldots, N-2.$$

Set $u^{N-1} := -(b_{N-1} + b_N + b)/2 = (1-b)/2$, $u^N = -b_N$ and $v^{N-1} := (1, 1, \ldots, 1)$ with $N-1$. $M := N(N+1)/2$ and define $u := [u^1, \ldots, u^N]$ and $v := [v^1, \ldots, v^{N-1}]$.

Theorem (Bostan and Y., 2020)

It holds that

$$\mathrm{Diag}((1+x_1)^{b_1}(1+x_1+x_2)^{b_2}\cdots(1+x_1+\cdots+x_N)^{b_N}(1+x_1+\cdots+2\,x_{N-1})^{b}) = {}_MF_{M-1}(u; v; (-N)^N t).$$

Example

For $N = 3$ and $R = b_1$, $b = S$, $b_2 = 0$, $b_3 = -1$:

$$\mathrm{Diag}\left((1+x)^R(1+x+2\,y)^S(1+x+y+z)^{-1}\right) =$$

$${}_4F_3\left(\left[\frac{1-(R+S)}{3}, \frac{2-(R+S)}{3}, \frac{3-(R+S)}{3}, \frac{1-S}{2}\right]; \left[\frac{1-(R+S)}{2}, \frac{2-(R+S)}{2}, 1\right]; -27t\right).$$

generalizes and explains [AKM, 2020]

$${}_3F_2\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{7}{9}\right], \left[1, \frac{2}{3}\right]; 27\,t\right) = \mathrm{Diag}\left(\frac{(1-x-2\,y)^{2/3}}{1-x-y-z}\right)$$

and

$${}_3F_2\left(\left[\frac{2}{9}, \frac{5}{9}, \frac{8}{9}\right], \left[1, \frac{5}{6}\right]; 27\,t\right) = \mathrm{Diag}\left(\frac{(1-x-2\,y)^{1/3}}{1-x-y-z}\right).$$