# Hypergeometric diagonals and a step towards Christol's conjecture[1]

Journées du GDR EFI, Toulouse

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[1] Joint work with Alin Bostan (doi.org/10.1090/proc/15693)

# A motivating example

Consider $f(x,y) := \sqrt{1-x}/(1-x-y) = \sum_{i,j\geq 0} f_{i,j} x^i y^j$.

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f(x,y) \;=\; \begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \iddots \\
x^0y^3 & \frac{7}{2}x^1y^3 & \frac{63}{8}x^2y^3 & \frac{231}{16}x^3y^3 & \cdots \\
x^0y^2 & \frac{5}{2}x^1y^2 & \frac{35}{8}x^2y^2 & \frac{105}{16}x^3y^2 & \cdots \\
x^0y^1 & \frac{3}{2}x^1y^1 & \frac{15}{8}x^2y^1 & \frac{35}{16}x^3y^1 & \cdots \\
x^0y^0 & \frac{1}{2}x^1y^0 & \frac{3}{8}x^2y^0 & \frac{5}{16}x^3y^0 & \cdots
\end{array}
$$

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We also have that

$${}_2F_1\left(\left[\frac{1}{4},\frac{3}{4}\right];\left[\frac{1}{2}\right];4t\right) = \sqrt{\frac{1+\sqrt{1-4t}}{2-8t}} = 1 + \frac{3}{2}t + \frac{35}{8}t^2 + \frac{231}{16}t^3 + \cdots.$$

# A motivating example

**Proof.**

$$\mathrm{Diag}(f) = [x^0]f(x, t/x) = \frac{1}{2\pi i}\oint \frac{f(x, t/x)}{x}dx.$$

```
> with(DEtools):
> F:=sqrt(1-x)/(1-x-y);
> G:=normal(1/x*subs(y=t/x,F));
> Zeilberger(G, t, x, Dt)[1];
```

Gives that $h = \mathrm{Diag}(f)$ satisfies $(16t^2 - 4t)h''(t) + (32t - 2)h'(t) + 3h(t) = 0.$ $\square$

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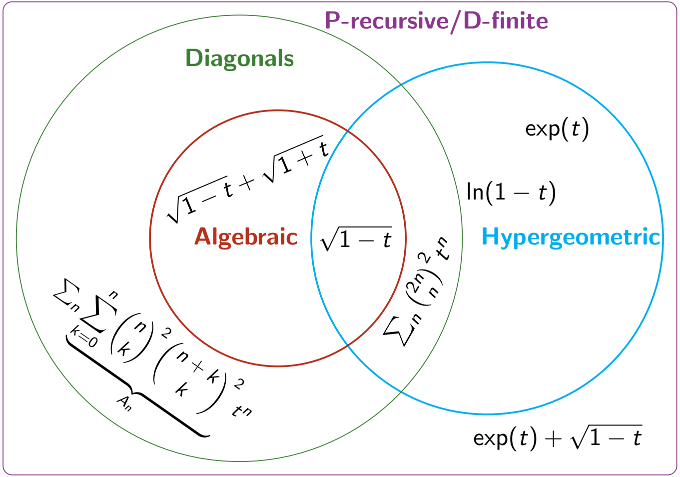
- Investigate general setting.
- Find and classify similar identities.
- “Inverse” creative telescoping.

# Definitions and interactions




A sequence $(u_n)_{n\geq 0}$ is **P-recursive**, if it satisfies a linear recurrence with polynomial coefficients:

$$c_d(n)u_{n+d} + \cdots + c_0(n)u_n = 0.$$

$(u_n)_{n\geq 0}$ is **hypergeometric** if $d = 1$.

Let $(\alpha)_n = \alpha\cdot(\alpha+1)\cdots(\alpha+n-1)$.

Then $u_n = \frac{(a)_n\cdot(b)_n}{(c)_n\cdot n!}$ satisfies

$$(c+n)(n+1)u_{n+1} - (a+n)(b+n)u_n = 0.$$

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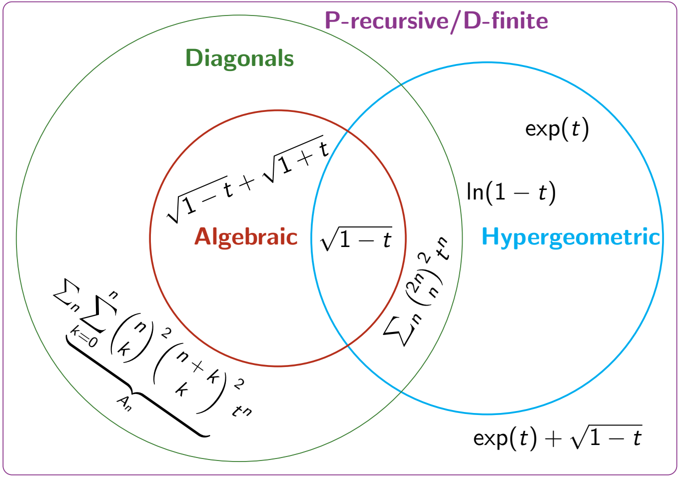


A power series $f(t) \in \mathbb{Q}[\![t]\!]$ is called **D-finite** if it satisfies a linear differential equation with polynomial coefficients:

$$p_n(t)f^{(n)}(t) + \cdots + p_0(t)f(t) = 0.$$

Let $(\alpha)_n = \alpha\cdot(\alpha+1)\cdots(\alpha+n-1)$.

Then ${}_2F_1\left[\begin{smallmatrix} a & b \\ & c \end{smallmatrix}; t\right] := \sum_{n\geq 0} \frac{(a)_n\cdot(b)_n}{(c)_n\cdot n!} t^n$ satisfies

$$t(1-t)f''(t)+(c-(a+b+1)t)f'(t)-abf(t) = 0.$$

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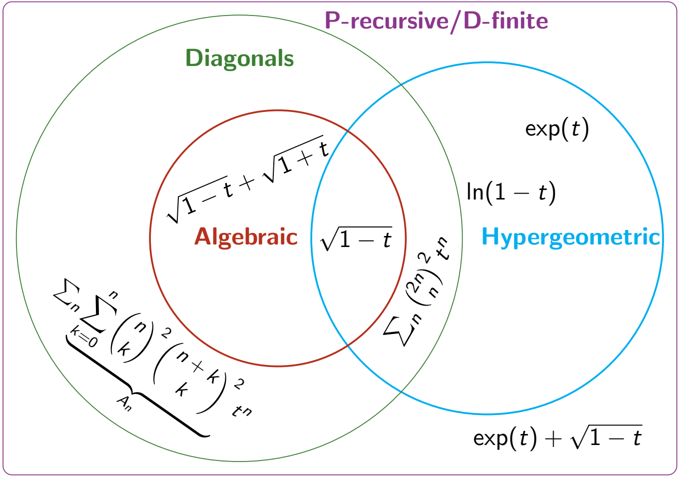


For a multivariate power series

$$f(x_1,\ldots,x_n) = \sum_{j_1,\ldots,j_n} f_{j_1,\ldots,j_n} x_1^{j_1}\cdots x_n^{j_n}$$

the diagonal is given by

$$\mathrm{Diag}(f) = \sum_j f_{j,j,\ldots,j} t^j \in \mathbb{Q}[\![t]\!].$$

**Diagonals** are series which can be written as diagonals of multivariate *rational* (equivalently **algebraic** [Denef,Lipshitz, 1987]) functions.

$$\mathrm{Diag}\left(\frac{1}{1-x-y}\right) = \mathrm{Diag}\sum_{i,j}\binom{i+j}{j}x^i y^j = \sum_n \binom{2n}{n} t^n = (1-4t)^{-1/2}$$

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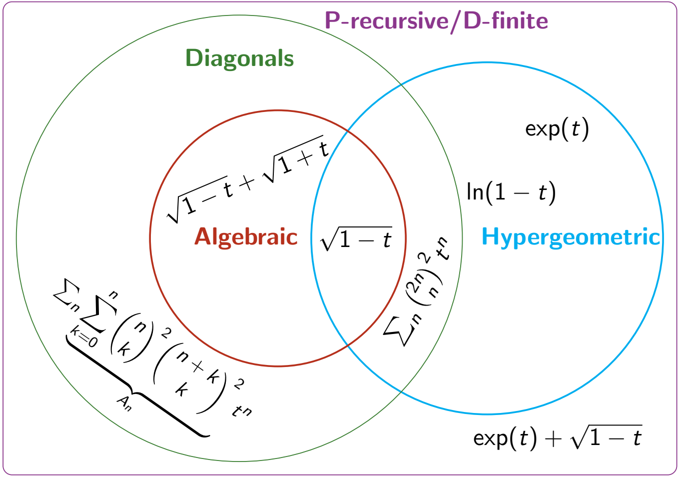


[Abel, 1827]:
**Algebraic** $\subseteq$ **D-finite**.

[Furstenberg, 1967]:
**Algebraic** $\subseteq$ **Diagonals**.

[Christol, 1984 and Lipshitz, 1988]:
**Diagonals** $\subseteq$ **D-finite**.

[Beukers, Heckman, 1989]:
**Algebraic** $\cap$ **Hypergeometric**.

[Bostan, Lairez, Salvy, 2017]:
**Diagonals** $=$ **Multiple binomial sums**.

Christol's Conjecture [Christol, 1987]: A convergent **D-finite** power series with **integer coefficients** is a **diagonal**.

# Globally bounded series

$f(t) \in \mathbb{Q}[\![t]\!]$ is globally bounded if $f$ has non-zero radius of convergence and there exist $\alpha, \beta \in \mathbb{N}^*$ such that $\alpha \cdot f(\beta \cdot t) \in \mathbb{Z}[\![t]\!]$.

**Proposition (Eisenstein's theorem)**

*If $g(x_1, \ldots, x_n) \in \mathbb{Q}[\![x_1, \ldots, x_n]\!]$ rational or algebraic, then $\mathrm{Diag}(g)$ is globally bounded.*

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$$f(t) = \sqrt{\frac{1+\sqrt{1-4t}}{2-8t}} = 1 + \frac{3}{2}t + \frac{35}{8}t^2 + \frac{231}{16}t^3 + \cdots \notin \mathbb{Z}[\![t]\!],$$

but

$$f(4\,t) = \sqrt{\frac{1+\sqrt{1-16t}}{2-32t}} = 1 + 6t + 70t^2 + 924t^3 + \cdots \in \mathbb{Z}[\![t]\!].$$

# Christol's conjecture(s)

$(C)$ If a power series $f \in \mathbb{Q}[\![t]\!]$ is **D-finite** and **globally bounded** then $f$ is a **diagonal**.

$(C')$ If a **hypergeometric function** and **globally bounded** then it is a **diagonal**.

$(C'')$ The function

$$ {}_3F_2\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\right], \left[\frac{1}{3}, 1\right]; 729\, t\right) = 1 + 60\, t + 20475\, t^2 + 9373650\, t^3 + \cdots $$

is a **diagonal**.

Clearly $(C) \Rightarrow (C') \Rightarrow (C'')$, but all open.

# More on hypergemetric functions

Let $(x)_j := x(x+1)\cdots(x+j-1)$ be the rising factorial. The hypergeometric function ${}_pF_q$ with rational parameters $a_1,\dots,a_p$ and $b_1,\dots,b_q$ is the univariate power series

$${}_pF_q([a_1,\dots,a_p],[b_1,\dots,b_q];t) := \sum_{j\geq 0} \frac{(a_1)_j\cdots(a_p)_j}{(b_1)_j\cdots(b_q)_j}\frac{t^j}{j!} \in \mathbb{Q}[\![t]\!].$$

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$$ {}_pF_q([a_1, \ldots, a_p], [b_1, \ldots, b_q]; t) := \sum_{j \geq 0} \frac{(a_1)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \frac{t^j}{j!} \in \mathbb{Q}[\![t]\!]. $$

- ${}_pF_q$ is not a polynomial and globally bounded $\Rightarrow q = p - 1$.
- Elegant criterion for testing whether a hypergeometric function is globally bounded or **algebraic** [Christol, 1986], [Beukers, Heckman, 1989].

$(C''*)$ List of 116 ${}_3F_2$'s which are potential counterexamples to Christol's conjecture [Bostan, Boukraa, Christol, Hassani, Maillard, 2011]:

$$ \mathrm{BBCHM} = \{ {}_3F_2([1/3, 5/9, 8/9], [1/2, 1]; t), {}_3F_2([1/4, 3/8, 5/6], [2/3, 1]; t), \ldots, \\ \ldots, {}_3F_2([1/9, 4/9, 5/9], [1/3, 1]; t), \ldots \}. $$

- Hypergeometric functions are excellent for testing Christol's conjecture.

# Hypergeometric functions: Algebraic or globally bounded?

**Theorem (Christol, 1986 and Beukers-Heckman, 1989)**

*Assume that the rational parameters* $\{a_1, \ldots, a_p\}$ *and* $\{b_1, \ldots, b_{p-1}, b_p = 1\}$ *are disjoint modulo* $\mathbb{Z}$. *Let* $N$ *be their common denominator. Then*

$$_pF_{p-1}([a_1, \ldots, a_p], [b_1, \ldots, b_{p-1}]; t) \qquad \textit{is}$$

- algebraic *if and only if for all* $1 \leq r < N$ *with* $\gcd(r, N) = 1$ *the numbers* $\{\exp(2\pi i r a_j), 1 \leq j \leq p\}$ *and* $\{\exp(2\pi i r b_j), 1 \leq j \leq p\}$ *interlace on the unit circle.*
- globally bounded *if and only if for all* $1 \leq r < N$ *with* $\gcd(r, N) = 1$, *one encounters more numbers in* $\{\exp(2\pi i r a_j), 1 \leq j \leq p\}$ *than in* $\{\exp(2\pi i r b_j), 1 \leq j \leq p\}$ *when running through the unit circle from* $1$ *to* $\exp(2\pi i)$.

# Interlacing criterion in practice

- Take $f(t) = {}_3F_2([1/4, 3/8, 7/8], [1/3, 2/3]; t)$. Is $f(t)$ **algebraic**?

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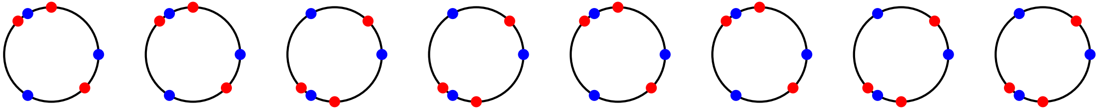

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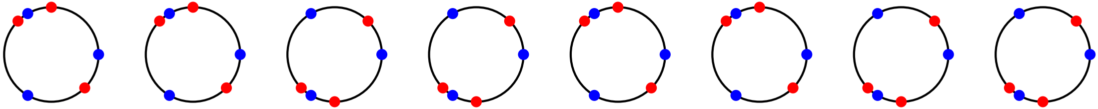

$\Rightarrow f(t)$ is **algebraic**.

# Result of Abdelaziz, Koutschan and Maillard, 2020

List of 116 ${}_3F_2$'s which are potential counterexamples to Christol's conjecture [Bostan, Boukraa, Christol, Hassani, Maillard, 2011]:

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$${}_3F_2\left(\left[\frac{1}{9}, \frac{4}{9}, \frac{7}{9}\right], \left[\frac{1}{3}, 1\right]; 27\,t\right) = \mathrm{Diag}\left(\frac{(1-x-y)^{2/3}}{1-x-y-z}\right), \quad \text{and}$$

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These identities [Abdelaziz, et al., 2020] solve two cases of $\mathrm{BBCHM}$. More generally,

$${}_3F_2\left(\left[\frac{1-R}{3}, \frac{2-R}{3}, \frac{3-R}{3}\right], [1, 1-R]; 27\,t\right) = \mathrm{Diag}\left(\frac{(1-x-y)^R}{1-x-y-z}\right),$$

for all $R \in \mathbb{Q}$. **Proof:** expand the rhs and use creative telescoping to simplify sums.

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$${}_3F_2\left(\left[\frac{1-R}{3}, \frac{2-R}{3}, \frac{3-R}{3}\right], [1, 1-R]; 27\,t\right) = \mathrm{Diag}\left((1-x-y)^R(1-x-y-z)^{-1}\right),$$

for all $R \in \mathbb{Q}$.

# Decidability

(i) Given an algebraic $g(x_1, \ldots, x_n)$ one can algorithmically decide whether $f(t) = \mathrm{Diag}(g)$ is hypergeometric:

- Creative telescoping: differential equation for $f(t)$.
- Recurrence relation for $(f_n)_{n\geq 0}$.
- Petkovšek's algorithm finds all hypergeometric solutions [Petkovšek, 1992].

(ii) Given a hypergeometric $f(t)$, finding an algebraic $g(x_1, \ldots, x_n)$ with $\mathrm{Diag}(g) = f(t)$ is completely open.

# Finding generalizations to $\mathrm{Diag}\left((1-x-y)^R(1-x-y-z)^{-1}\right)$

Try $f(t) = \mathrm{Diag}((1-x-y)^{1/3}(1-x-y-z)^{-1/2}) = 1 + \frac{91}{72}t + \frac{191425}{13824}t^2 + \cdots$.

- Creative telescoping or guessing implies/suggests that

$$91\,f + (44084t-72)\,f' + 216t(698t-9)\,f'' + 72t^2(1242t-31)\,f''' + 432t^3(27t-1)\,f^{(iv)} = 0$$

- It follows that

$$\frac{f_{n+1}}{f_n} = \frac{(18n+7)\,(2n+1)\,(18n+1)\,(18n+13)}{72\,(6n+1)\,(n+1)^3}$$

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Try $f(t)=\operatorname{Diag}((1-x-y)^{1/3}(1-x-y-z-w)^{-1})=1+\frac{176}{9}t+\frac{54740}{27}t^2+\cdots$.

- Guessing suggests that

$$\frac{f_{n+1}}{f_n}=\frac{8\,(12n+11)\,(3n+2)\,(2n+1)\,(12n+5)\,(6n+1)}{9\,(6n+5)\,(3n+1)\,(n+1)^3}$$

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$$\frac{f_{n+1}}{f_n} = \frac{(6n+5)(18n+1)(18n+13)(3n+1)(18n+7)}{9(3n+2)(12n+1)(12n+7)(n+1)^2}$$

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- $f(t)$ is hypergeometric!

Try $f(t) = \mathrm{Diag}((1-x-y)^{-1}(1-x-z)^{-1}(1-y-z)^{-1}) = 1 + 14t + 145t^2 + \cdots$.

- Guessing suggests that

$$2n(5n-4)(2n+1)\, f_n = (295n^3 - 156n^2 - 61n + 6)\, f_{n-1} + 24(5n+1)(3n-1)(3n-2)\, f_{n-2}$$

- $f(t)$ is **not** hypergeometric!

# Main result

**Theorem (Bostan and Y., 2021)**

*Let* $N \in \mathbb{N} \setminus \{0\}$ *and* $b_1, \dots, b_N \in \mathbb{Q}$ *with* $b_N \neq 0$*. Then*

$$\mathrm{Diag}((1-x_1)^{b_1}(1-x_1-x_2)^{b_2} \cdots (1-x_1-\cdots-x_N)^{b_N})$$

*is a* hypergeometric function.

# Complete identity

Let $B(k) := -(b_k + \cdots + b_N)$.

$$u^k := \left( \frac{B(k)}{N-k+1}, \frac{B(k)+1}{N-k+1}, \dots, \frac{B(k)+N-k}{N-k+1} \right), \quad k = 1, \dots, N,$$

$$v^k := \left( \frac{B(k)}{N-k}, \frac{B(k)+1}{N-k}, \dots, \frac{B(k)+N-k-1}{N-k} \right), \quad k = 1, \dots, N-1.$$

Set $v^N := (1, 1, \dots, 1)$ with $N-1$ ones, $u := [u^1, \dots, u^N]$ and $v := [v^1, \dots, v^N]$.

**Theorem (Bostan and Y., 2020)**

*Let $N \in \mathbb{N} \setminus \{0\}$ and $b_1, \dots, b_N \in \mathbb{Q}$ with $b_N \neq 0$. Then*

$$\mathrm{Diag}((1-x_1)^{b_1}(1-x_1-x_2)^{b_2} \cdots (1-x_1-\cdots-x_N)^{b_N})$$

*is* hypergeometric *and equal to*

$$ {}_{\binom{N+1}{2}}F_{\binom{N+1}{2}-1}(u; v; N^N t).$$

# Examples

- If $N = 2$ we have

$$\mathrm{Diag}\left((1-x)^R(1-x-y)^S\right) = {}_3F_2\left(\left[\frac{-(R+S)}{2}, \frac{-(R+S)+1}{2}, -S\right]; [-(R+S), 1]; 4t\right).$$

- Hence $\mathrm{Diag}((1-x)^{1/2}(1-x-y)^{-1}) = {}_2F_1\left(\left[\frac{1}{4}, \frac{3}{4}\right]; \left[\frac{1}{2}\right]; 4t\right)$.
- Letting $N = 3$ we obtain

$$\mathrm{Diag}\left((1-x)^R(1-x-y)^S(1-x-y-z)^T\right) =$$

$$ {}_6F_5\left(\left[\frac{-(R+S+T)}{3}, \frac{-(R+S+T)+1}{3}, \frac{-(R+S+T)+2}{3}, \frac{-(S+T)}{2}, \frac{-(S+T)+1}{2}, -T\right];\right.$$
$$\left.\left[\frac{-(R+S+T)}{2}, \frac{-(R+S+T)+1}{2}, -(S+T), 1, 1\right]; 27t\right).$$

- Hence $\mathrm{Diag}\left(\frac{(1-x-y)^S}{1-x-y-z}\right) = {}_3F_2\left(\left[\frac{1-S}{3}, \frac{2-S}{3}, \frac{3-S}{3}\right], [1, 1-S]; 27\,t\right)$.

# Sketch of the proof

**Proof.**

$$[x_1^{k_1} \cdots x_{N-1}^{k_{N-1}} \cdot x_N^{k_N}](1+x_1)^{b_1} \cdots (1+x_1+\cdots+x_{N-1})^{b_{N-1}}(1+x_1+\cdots+x_N)^{b_N}$$

$$=$$

$\square$

# Sketch of the proof

**Proof.**

$$[x_1^{k_1} \cdots x_{N-1}^{k_{N-1}} \cdot x_N^{k_N}](1+x_1)^{b_1} \cdots (1+x_1+\cdots+x_{N-1})^{b_{N-1}}(1+x_1+\cdots+x_N)^{b_N}$$

$$= \binom{b_N}{k_N}$$

$\square$

# Sketch of the proof

**Proof.**

$$[x_1^{k_1}\cdots x_{N-1}^{k_{N-1}}\cdot x_N^{k_N}](1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_{N-1})^{b_{N-1}}(1+x_1+\cdots+x_N)^{b_N}$$
$$=\binom{b_N}{k_N}\binom{b_{N-1}+b_N-k_N}{k_{N-1}}$$

□

## Sketch of the proof

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$$=\binom{b_N}{k_N}\binom{b_{N-1}+b_N-k_N}{k_{N-1}}\cdots\binom{b_1+\cdots+b_{N-1}+b_N-k_N-k_{N-1}-\cdots-k_2}{k_1}.$$

$\square$

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$$=\binom{b_N}{k_N}\binom{b_{N-1}+b_N-k_N}{k_{N-1}}\cdots\binom{b_1+\cdots+b_{N-1}+b_N-k_N-k_{N-1}-\cdots-k_2}{k_1}.$$

By definition:

$$[t^n]_M F_{M-1}(u;v;(-N)^N t)=(-1)^{Nn}N^{Nn}\frac{\prod_{i,j}(u_j^{(i)})_n}{\prod_{i,j}(v_j^{(i)})_n\cdot n!}.$$

$\square$

# Sketch of the proof

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$$[x_1^{k_1}\cdots x_{N-1}^{k_{N-1}}\cdot x_N^{k_N}](1+x_1)^{b_1}\cdots(1+x_1+\cdots+x_{N-1})^{b_{N-1}}(1+x_1+\cdots+x_N)^{b_N}$$
$$=\binom{b_N}{k_N}\binom{b_{N-1}+b_N-k_N}{k_{N-1}}\cdots\binom{b_1+\cdots+b_{N-1}+b_N-k_N-k_{N-1}-\cdots-k_2}{k_1}.$$

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Finally,

$$\binom{b_N}{n}\cdots\binom{b_1+\cdots+b_N-(N-1)n}{n}=(-1)^{Nn}N^{Nn}\frac{\prod_{i,j}(u_j^{(i)})_n}{\prod_{i,j}(v_j^{(i)})_n\cdot n!}.$$

$\square$

# Some corollaries

## Corollary

*Let $f(t) = \mathrm{Diag}((1+x_1)^{b_1} \cdots (1+x_1+\cdots+x_N)^{b_N})$, then $f$ is algebraic if and only if $N=2$ and $b_2 \in \mathbb{Z}$, or $N=1$.*

## Sketch of proof.

$$ {}_MF_{M-1}(u;v;t) = {}_MF_{M-1}([u^{(1)},\ldots,u^{(N)}];[v^{(1)},\ldots,v^{(N-1)},\underbrace{1,1,\ldots,1}_{N-1 \text{ times}}];t). $$

At most one cancellation between $u^{(k)}$ and a 1. By Christol's theorem, $N \leq 2$. □

# Some corollaries

## Corollary

*Let $f(t) = \mathrm{Diag}((1+x_1)^{b_1} \cdots (1+x_1+\cdots+x_N)^{b_N})$, then $f$ is algebraic if and only if $N = 2$ and $b_2 \in \mathbb{Z}$, or $N = 1$.*

The **Hadamard grade** of $f(t)$ is the least positive integer $n = n(f)$ such that $f(t)$ can be written as the Hadamard (term-wise) product of $n$ algebraic power series.

## Corollary

*The Hadamard grade of $\mathrm{Diag}((1+x_1)^{b_1} \cdots (1+x_1+\cdots+x_N)^{b_N})$ is finite and $\leq N$.*

# Some corollaries

## Corollary

*Let* $f(t) = \mathrm{Diag}((1+x_1)^{b_1} \cdots (1+x_1+\cdots+x_N)^{b_N})$*, then* $f$ *is algebraic if and only if* $N = 2$ *and* $b_2 \in \mathbb{Z}$*, or* $N = 1$*.*

The **Hadamard grade** of $f(t)$ is the least positive integer $n = n(f)$ such that $f(t)$ can be written as the Hadamard (term-wise) product of $n$ algebraic power series.

## Corollary

*The Hadamard grade of* $\mathrm{Diag}((1+x_1)^{b_1} \cdots (1+x_1+\cdots+x_N)^{b_N})$ *is finite and* $\leq N$*.*

## Sketch of proof.

$$ {}_M F_{M-1}(u; v; t) = {}_N F_{N-1}(u^{(1)}; v^{(1)}; t) \star {}_{N-1}F_{N-2}(u^{(2)}; v^{(2)}; t) \star \cdots \star {}_1F_0(u^{(N)};; t), $$

and each ${}_{N-k+1}F_{N-k}(u^{(k)}; v^{(k)}; t)$ is algebraic [Beukers, Heckman, 1989]. □

# More results and limitations

- From the list $\mathrm{BBCHM}$ (116 elements) we could resolve 40 cases, so 78 potential ${}_3F_2$ counterexamples to Christol's conjecture remain.

- Assuming the Rohrlich-Lang conjecture, [Rivoal, Roques, 2014] could prove that

$$ {}_3F_2\left(\left[\frac{1}{7},\frac{2}{7},\frac{4}{7}\right],\left[1,\frac{1}{2}\right],2401\,t\right) = 1 + 112\,t + 103488\,t^2 + 139087872\,t^3 + \cdots $$

has infinite Hadamard grade.

# Summary and conclusion

- The functions

$$_{N(N+1)/2}F_{N(N+1)/2-1}([u^{(1)}, \dots, u^{(N)}]; [v^{(1)}, \dots, v^{(N)}]; N^N t)$$

are **globally bounded** and (nice) **diagonals**.

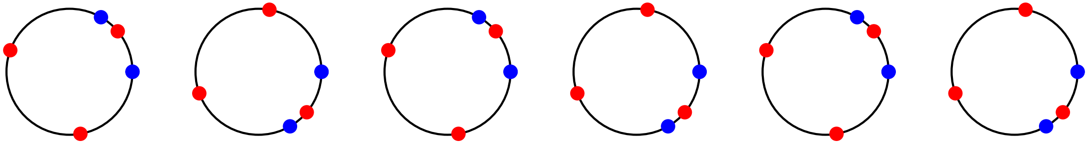
- The functions $\mathrm{Diag}((1-x_1)^{b_1} \cdots (1-x_1-\cdots-x_N)^{b_N})$ are **hypergeometric**.

- The main identities of [Abdelaziz,Koutschan,Maillard, 2020] fit in a larger picture.

- Christol's conjecture is still widely open, but we are getting (a bit) closer.

# Bonus: Globally bounded hypergeometric functions in practice

- Is $f(t) = {}_3F_2([1/9, 4/9, 5/9], [1/6, 1]; t)$ **globally bounded**?
- Common denominator of the parameters: $N = 18$.
- We have $\varphi(18) = 6$, and each $r \in \{1, 5, 7, 11, 13, 17\} =: S$ is coprime to 18.
- For each $r \in S$ we look at $\{\exp(2\pi ir \cdot 1/9), \exp(2\pi ir \cdot 4/9), \exp(2\pi ir \cdot 7/9)\}$ and $\{\exp(2\pi ir \cdot 1/6), \exp(2\pi ir \cdot 1), \exp(2\pi ir \cdot 1)\}$.



$\Rightarrow f(t)$ is **globally bounded**.

# Bonus: Main result II

**Theorem (Bostan and Y., 2020)**

*Let* $N \in \mathbb{N} \setminus \{0\}$ *and* $b_1, \ldots, b_N \in \mathbb{Q}$ *with* $b_N \neq 0$ *and* $b_{N-1} + b_N = -1$. *Then for any* $b \in \mathbb{Q}$,

$$\mathrm{Diag}((1+x_1)^{b_1}(1+x_1+x_2)^{b_2}\cdots(1+x_1+\cdots+x_N)^{b_N}\cdot(1+x_1+\cdots+2\,x_{N-1})^{b})$$

*is a hypergeometric function.*

# Bonus: Complete identity

Let $B(k) := -(b_k + \cdots + b_N + b)$.

$$u^k := \left(\frac{B(k)}{N-k+1}, \frac{B(k)+1}{N-k+1}, \ldots, \frac{B(k)+N-k}{N-k+1}\right), \quad k = 1, \ldots, N-2$$

$$v^k := \left(\frac{B(k)}{N-k}, \frac{B(k)+1}{N-k}, \ldots, \frac{B(k)+N-k-1}{N-k}\right), \quad k = 1, \ldots, N-2.$$

Set $u^{N-1} := -(b_{N-1} + b_N + b)/2 = (1-b)/2$, $u^N = -b_N$ and $v^{N-1} := (1, 1, \ldots, 1)$ with $N-1$. $M := N(N+1)/2$ and define $u := [u^1, \ldots, u^N]$ and $v := [v^1, \ldots, v^{N-1}]$.

**Theorem (Bostan and Y., 2020)**

*It holds that*

$$\mathrm{Diag}((1+x_1)^{b_1}(1+x_1+x_2)^{b_2}\cdots(1+x_1+\cdots+x_N)^{b_N}(1+x_1+\cdots+2\,x_{N-1})^{b}) \\ = {}_M F_{M-1}(u; v; (-N)^N t).$$