The generating function of DYZ-like numbers is algebraic[1]

FoCM23

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20th of June, 2023

[1] Joint work with Alin Bostan and Jacques-Arthur Weil.

Two sequences

$$(a_n)_{n\geq 0} = (1\,,\ -48300\,,\ 7981725900\,,\ -1469166887370000, \ldots)$$

$$(b_n)_{n\geq 0} = (1\,,\ -144900\,,\ 88464128725\,,\ -62270073456990000, \ldots)$$

Origin of a_n and b_n

- In Arithmetic and Topology of Differential Equations, 2018 by Don Zagier:

$$u_{n-3} + 20\left(4500n^2 - 18900n + 19739\right) u_{n-2} + 80352000n(5n-1)(5n-2)(5n-4)u_n + \\ +25\left(2592000n^4 - 16588800n^3 + 39118320n^2 - 39189168n + 14092603\right) u_{n-1} = 0,$$

with initial terms $u_0 = 1, u_1 = -161/(2^{10}\cdot 3^5)$ and $u_2 = 26605753/(2^{23}\cdot 3^{12}\cdot 5^2)$.

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Problem (Zagier, 2018)

Find $(\alpha, \beta) \in \mathbb{Q}^* \times \mathbb{Q}^*$ such that $u_n \cdot (\alpha)_n \cdot (\beta)_n \cdot \gamma^n \in \mathbb{Z}$ for some $\gamma \in \mathbb{Z}^*$.

$$(x)_n := x \cdot (x+1) \cdots (x+n-1).$$

- [Yang and Zagier]: $a_n = u_n \cdot (3/5)_n \cdot (4/5)_n \cdot (2^{10} \cdot 3^5 \cdot 5^4)^n \in \mathbb{Z}$,
- [Dubrovin and Yang]: $b_n = u_n \cdot (2/5)_n \cdot (9/10)_n \cdot (2^{12} \cdot 3^5 \cdot 5^4)^n \in \mathbb{Z}$.

Mystery of a_n and b_n

- "Yang and I found a formula showing that the numbers a_n are integers [...]"
  "Dubrovin and Yang found that the numbers b_n are *also* integral and that in this case the generating function [...] is actually **algebraic**!"
- "So this is a very mysterious example" – [Zagier, 2018]
- "My presumed arithmetic intuition [...] was entirely broken" – [Wadim Zudilin]

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Problem

Investigate the nature of $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ and similar sequences.

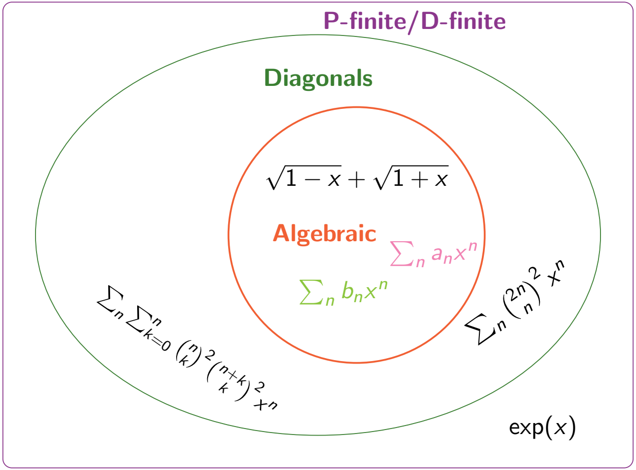
Theorem (Bostan, Weil, Y.)

The generating functions of both $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are **algebraic**.

Theorem (Bostan, Weil, Y.)

Seven more solutions to Zagier's problem: $(c_n)_{n\geq 0}, \ldots, (i_n)_{n\geq 0} \in \mathbb{Z}$.

Definitions and interactions



A sequence $(u_n)_{n\geq 0}$ is **P-finite**, if it satisfies a linear recurrence with polynomial coefficients:

$$c_r(n)u_{n+r}+\cdots+c_0(n)u_n=0.$$

$u_n=\binom{2n}{n}$ satisfies

$$(n+1)u_{n+1}-(2+4n)u_n=0.$$

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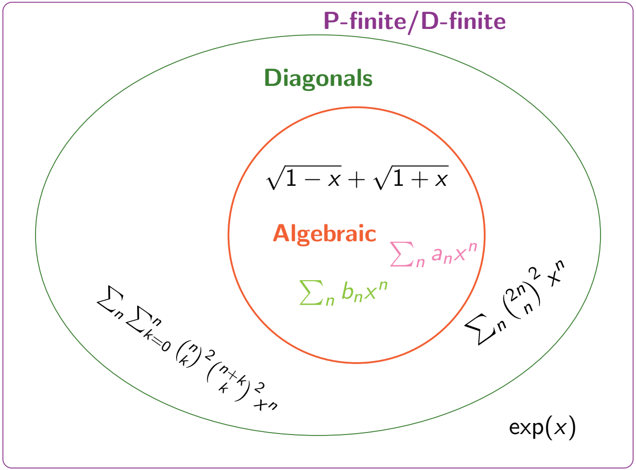

A power series $f(x) \in \mathbb{Q}[[x]]$ is **D-finite** if it satisfies a linear differential equation with polynomial coefficients:

$$p_n(x) f^{(n)}(x) + \cdots + p_0(x) f(x) = 0.$$

This equation can be rewritten: $L \cdot f = 0$,

$L = p_n(x)\partial^n + \cdots + p_0(x) \in \mathbb{Q}(x)[\partial]$,

where $\partial := \frac{\mathrm{d}}{\mathrm{dx}}$.

$\sqrt{1-x}+\sqrt{1+x}$ satisfies

$4(x^2-1)f''(x) + 4xf'(x) - f(x) = 0.$

$$L = 4(x^2-1)\partial^2 + 4x\partial - 1.$$

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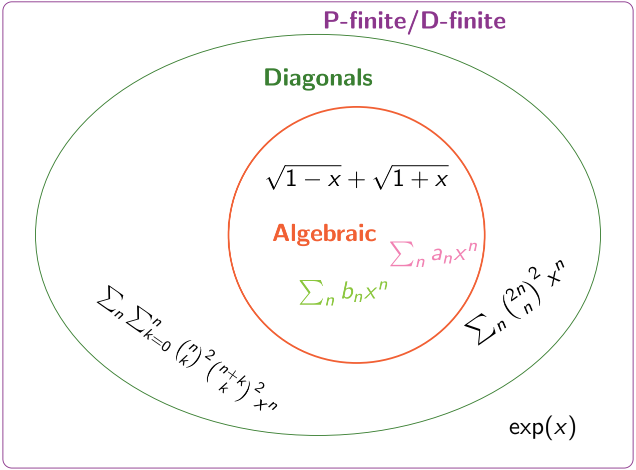

A series $f(x) \in \mathbb{Q}[[x]]$ is a **Diagonal** if there exists a rational function

$$R = \sum_{i_1,\dots,i_n \geq 0} c_{i_1,\dots,i_n} t_1^{i_1} \cdots t_n^{i_n} \in \mathbb{Q}(t_1,\dots,t_n)$$

such that

$$f(x) = \mathrm{Diag}(R) := \sum_{k \geq 0} c_{k,\dots,k} x^k.$$

$$\mathrm{Diag}\frac{1}{1-t_1-t_2} = \mathrm{Diag}\sum_{i,j\geq 0}\binom{i+j}{j} t_1^i t_2^j$$

$$= \sum_{k\geq 0}\binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}$$

Back to a_n and b_n

- $(a_n)_n$ and $(b_n)_n$ are **P-finite** sequences $\Rightarrow$ generating functions are **D-finite**.

$$L_a = 1800x(7x-62)\left(x^2+50x+20\right)\partial^2 + 720(42x^3+173x^2-14230x-620)\partial + 6048x^2 - 139453x - 249550 \in \mathbb{Q}(x)[\partial],$$

$$\begin{aligned} L_b = {} & 90000x^3(2911x+310)\left(x^2+50x+20\right)\partial^4 \\ & + 18000x^2\left(154283x^3+5185005x^2+1675710x+142600\right)\partial^3 \\ & + 50x\left(147290778x^3+2740219655x^2+566777510x+37497600\right)\partial^2 \\ & + 5\left(919899288x^3+5629046605x^2+1348939210x+10713600\right)\partial \\ & + 18\left(13937868x^2-1076845x+1247750\right) \in \mathbb{Q}(x)[\partial]. \end{aligned}$$

- The generating functions of $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ solve $L_a \cdot y = 0$ and $L_b \cdot y = 0$.

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Stanley's problem (1980)

Given a **D-finite** series, how to prove or disprove that it is **algebraic**?

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- **Disproving** **algebraicity** often easier in practice [Flajolet, 1987], [Bostan, 2017].
- Tests for justifying **algebraicity** based on **conjectures** or **numerics**:
  - **Grothendieck-Katz** conjecture (integrality of coefficients $\leftrightarrow$ algebraic solutions)
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- Applied differential Galois theory can prove **algebraicity** in practice.

Proving transcendence of D-finite functions

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Some useful properties of algebraic functions $f(x) = \sum_{n\geq 0} u_n x^n$:

1. Coefficient sequence is globally bounded: $u_n w^n \in \mathbb{Z}$. $\log(1-x) \notin \overline{\mathbb{Q}(x)}$
2. Special asymptotics: $u_n = \frac{\rho^n n^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^n C_i \omega_i^n + O(\rho^n n^\beta)$. $\sum_{n\geq 0} \binom{2n}{n}^2 x^n \notin \overline{\mathbb{Q}(x)}$
3. Evaluation at algebraic numbers: $f(\alpha) \in \overline{\mathbb{Q}}$ for $\alpha \in \overline{\mathbb{Q}}$. $\exp(x) \notin \overline{\mathbb{Q}(x)}$
4. Minimal ODE has basis of solutions with no log's. $\sum_{n\geq 0}\sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2 x^n \notin \overline{\mathbb{Q}(x)}$

André-Christol conjecture: 1 + 4 also sufficient.

Differential Galois theory: proving algebraicity

- $L \cdot y = 0$ is equivalent to $Y' = A(x)Y$, where $A(x) \in M^{n\times n}(k)$ and $k = \overline{\mathbb{Q}}(x)$.
- Picard-Vessiot extension: $K = k(U)$, where U is a fundamental solution matrix.
- The differential Galois group G is the group of field automorphisms of K which commute with the derivation and leave all elements of k invariant:

$$G := \mathrm{Aut}_\partial(K/k) = \{\sigma \in \mathrm{Aut}(K) \colon \sigma|_k \equiv \mathrm{id}_k \text{ and } \sigma \circ \partial \equiv \partial \circ \sigma\}.$$

- G is a linear algebraic subgroup of $\mathrm{GL}_n(\overline{\mathbb{Q}})$.
- G stabilizes the ideal of differential relations between solutions.

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- In practice G is very difficult to compute [Hrushovski, 2002], [Feng, 2015], [van der Hoeven, 2007], [Amzallag, Minchenko, Pogudin, 2018], [Sun, 2019].

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- Theory and algorithm for computing $\mathfrak{g}$ [Barkatou, Cluzeau, Di Vizio, Weil, 2020].
- Idea: Compute symmetric powers of L and find **rational solutions** of them. These solutions yield information for $\mathfrak{g}$ via solving a **linear** system.

Toy example

- The operator $L = (4x^2 - 4)\partial^2 + 4x\partial - 1$ has a basis of algebraic solutions:
$$\sqrt{1+x} + \sqrt{1-x} \text{ and } \sqrt{1+x} - \sqrt{1-x}.$$
- $L \cdot y = 0$ is equivalent to $Y' = A(x)Y$ where $A(x) = \begin{pmatrix} 0 & 1 \\ \frac{1}{4x^2-4} & \frac{-4x}{4x^2-4} \end{pmatrix}$.

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- The operator $L = (4x^2 - 4)\partial^2 + 4x\partial - 1$
- $L \cdot y = 0$ is equivalent to $Y' = A(x)Y$ where $A(x) = \begin{pmatrix} 0 & 1 \\ \frac{1}{4x^2-4} & \frac{-4x}{4x^2-4} \end{pmatrix}$.
- If $Y = (y_1, y_2)^t$ is a solution to $Y' = A(x)Y$ then $Y = (y_1^2, 2y_1y_2, y_2^2)^t$ is a solution to the symmetric square system $Y' = A^{(2)}(x)Y$, where now

$$A^{(2)}(x) = \frac{1}{4(x^2-1)} \begin{pmatrix} 0 & 4x^2-4 & 0 \\ 2 & -4x & 8x^2-8 \\ 0 & 1 & -8x \end{pmatrix}.$$

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- It has rational solutions! $F_1 = (4x, 4, x/(x^2-1))^t$, $F_2 = (-4, 0, 1/(x^2-1))^t$.
- If $M \in \mathfrak{g}^{(2)}$ then $MF = 0$ and M comes from a symmetric square. I.e. M satisfies

$$\begin{pmatrix} 2m_{1,1} & m_{1,2} & 0 \\ 2m_{2,1} & m_{1,1} + m_{2,2} & 2m_{1,2} \\ 0 & m_{2,1} & 2m_{2,2} \end{pmatrix} \cdot F_\ell = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad m_{i,j} \in \mathbb{Q}(x), \ell = 1, 2.$$

- The only solution is $m_{i,j} = 0$. Hence $\mathfrak{g}^{(2)} = \mathfrak{g} = 0$. All solutions of L are algebraic.

The generating sequence of $(b_n)_n$ is algebraic (known to Dubrovin & Yang)

- For L_b same method as in the toy example works.
- $L_b \cdot y = 0$ equivalent to $Y' = A(x)Y$ for $A(x) \in M^{4\times 4}(\mathbb{Q}(x))$.
- The fifth symmetric power $Y' = A^{(5)}(x)Y$ has rational solutions.

```
> A5 := SymmetricPowerSystem(A,5):
```

- $A^{(5)}(x) \in M^{N\times N}(\mathbb{Q}(x))$, where $N = \binom{4+5-1}{4-1} = 56$.
- Finding the rational solutions takes ≈2 min on a regular PC.

```
> V:=RationalSolutions([A5],[x]):
```

- The corresponding system in $m_{i,j}$ has no non-zero solutions in $\mathbb{Q}(x)$ (≈15 sec).

```
> G:=Matrix(4,4,symbol=g):
> G5 := SymmetricPowerSystem(G,5):
> sol := solve(convert(G5.V, set)):
```

- sol $= 0 \Rightarrow \mathfrak{g}_b = 0$, therefore L_b has only algebraic solutions.

The generating sequence of $(a_n)_n$ is algebraic (new)

- For the generating function of $(a_n)_{n\geq 0}$ same method as for $(b_n)_{n\geq 0}$ works.
- The 20th symmetric power of L_a has rational solutions (≈4 sec).

```
> A20 := SymmetricPowerSystem(A,20):
> V:=RationalSolutions([A20],[x]):
```

- $A^{(20)} \in M^{N\times N}(\mathbb{Q}(x))$, where $N = \binom{2+20-1}{2-1} = 21$.
- The corresponding system in $m_{i,j}$ has no non-zero solutions in $\mathbb{Q}(x)$ (≈0.4 sec).

```
> G:=Matrix(2,2,symbol=g):
> G20 := SymmetricPowerSystem(G,20):
> sol := solve(convert(G20.V, set)):
```

- $\mathtt{sol} = 0 \Rightarrow \mathfrak{g}_a = 0$, therefore L_a has only algebraic solutions.

DYZ-like numbers

Zagier's problem

Find $(\alpha, \beta) \in \mathbb{Q}^* \times \mathbb{Q}^*$ such that $u_n \cdot (\alpha)_n \cdot (\beta)_n \cdot \gamma^n \in \mathbb{Z}$ for some $\gamma \in \mathbb{Z}^*$.

$$(x)_n := x \cdot (x+1) \cdots (x+n-1).$$

#	u	v	ODE order	degree	#	u	v	ODE order	degree
a_n	3/5	4/5	2	120	f_n	19/60	49/60	4	155520
b_n	2/5	9/10	4	120	g_n	19/60	59/60	4	46080
c_n	1/5	4/5	2	120	h_n	29/60	49/60	4	46080
d_n	7/30	9/10	4	155520	i_n	29/60	59/60	4	155520
e_n	9/10	17/30	4	155520					

Theorem (Bostan, Weil, Y., 2023)

The sequences $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}, (c_n)_{n\geq 0}, \ldots, (i_n)_{n\geq 0}$ are solutions to Zagier's problem.

- Estimates for degrees based on numerical monodromy group computations.
- Proof of **algebraicity**: Done: a_n, b_n, c_n. In progress: $d_n, e_n, f_n, g_n, h_n, i_n$.

Summary

- Both sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ have algebraic generating functions.
- Seven more sequences are solutions to Zagier's problem.
- Differential Galois theory allows efficient proving that **D-finite** series is **algebraic**.

Bonus: explicit solution for $\sum_{n\geq 0} a_n x^n$

We saw that $\sum_{n\geq 0} a_n x^n$ is a solution of

$$q_2(x)y''(x) + q_1(x)y'(x) + q_0(x)y(x) = 0, \quad \text{where} \tag{1}$$

$$q_2(x) = 5x(302400x - 31)(373248000x^2 + 216000x + 1),$$

$$q_1(x) = 1354442342400000x^3 + 64571904000x^2 - 61473600x - 31,$$

$$q_0(x) = 300(902961561600x^2 - 240974784x - 4991).$$

Maple's `dsolve(deq)` shows that every solution of (1) is a linear combination of

$$u_1(x) \cdot {}_2F_1\left[\begin{matrix} -1/60 & 11/60 \\ \multicolumn{2}{c}{2/3} \end{matrix}; \frac{p_1(x)}{p_2(x)}\right] \quad \text{and} \quad u_2(x) \cdot {}_2F_1\left[\begin{matrix} 19/60 & 31/60 \\ \multicolumn{2}{c}{4/3} \end{matrix}; \frac{p_1(x)}{p_2(x)}\right],$$

where ${}_2F_1\left[\begin{matrix} a & b \\ \multicolumn{2}{c}{c} \end{matrix}; x\right]$ is the Gaussian hypergeometric function

$${}_2F_1\left[\begin{matrix} a & b \\ \multicolumn{2}{c}{c} \end{matrix}; x\right] := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad (u)_j := u(u+1)\cdots(u+j-1).$$

Bonus: Integrality of the sequences

- Recall:

$$u_{n-3} + 20\left(4500n^2 - 18900n + 19739\right)u_{n-2} + 80352000n(5n-1)(5n-2)(5n-4)u_n +$$
$$+25\left(2592000n^4 - 16588800n^3 + 39118320n^2 - 39189168n + 14092603\right)u_{n-1} = 0,$$

with initial terms $u_0 = 1$, $u_1 = -161/(2^{10}\cdot 3^5)$ and $u_2 = 26605753/(2^{23}\cdot 3^{12}\cdot 5^2)$.

- In [Dubrovin, Yang, Zagier, 2022] proven:

$$u_n = 6^{-5n}\cdot \sum_{s=0}^{5n/2} \frac{(-9)^s}{10^{2s}}\cdot \frac{\left(\frac{1}{5}\right)_{3n-s}}{s!(5n-2s)!}.$$

- Count primes *à la* Legendre:

$$\frac{\left(\frac{1}{5}\right)_{3n-s}(\alpha)_n(\beta)_n}{s!(5n-2s)!} \in \mathbb{Z}[1/30] \quad \text{for } s, n \in \mathbb{N},$$

and for (α, β) in the presented table.

Bonus: Origin of $(c_n)_{n\geq 0}$

- For a simple Lie-algebra $(\mathfrak{g}, [\cdot,\cdot])$ [Bertola, Dubrovin, Yang, 2015] define the so-called *topological ordinary differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} M = [M, \Lambda],$$

where $M = M(\lambda)$ and $\Lambda = I_+ + \lambda E_{-\theta}$, for a principal nilpotent element $I_+ = \sum_{i=1}^{n} E_i$ and (normalized) $E_{-\theta} \in \mathfrak{g}_{-\theta}$.

- For $\mathfrak{g} = \mathrm{sl}_{n+1}(\mathbb{C})$ one finds

$$\Lambda = \begin{pmatrix} 0 & I_n \\ \lambda & 0 \end{pmatrix}, \quad I_n \text{ is the } n \times n \text{ identity matrix.}$$

and the (normalized) (dominant) ODE reads

$$64800000x^3(x+155)y^{(iv)}(x) + (x^2 - 1220x + 623875)y(x) + 7200(10x^2 + 3209x + 133920)y'(x) + 18000x(5x^2 + 6091x + 1874880)y''(x) + 12960000x^2(18x+3565)y'''(x) = 0$$

- Then $\sum_{n\geq 0} c_n x^n$ is the unique power series solution.