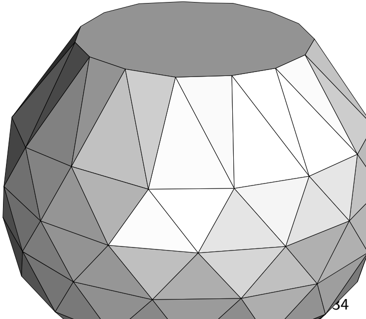
On Rupert's problem[1]

Math Colloquium at University of Vienna

Jakob Steininger and Sergey Yurkevich

28th of January, 2026

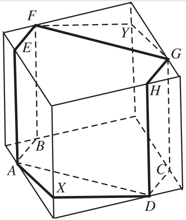


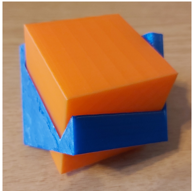
[1] Based on arxiv.org/abs/2508.18475 and arxiv.org/abs/2112.13754

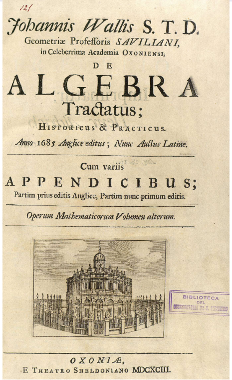
Prince Rupert's cube

Fact (Wallis, 1685)

It is possible to cut a hole inside the **unit cube** such that another **unit cube** can pass through this hole.






121

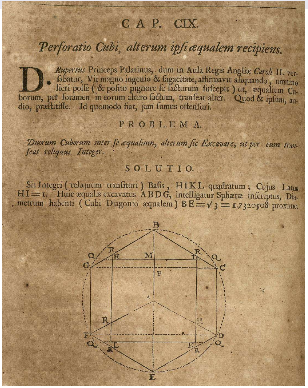
Johannis Wallis S. T. D.
Geometriæ Profeſſoris *SAVILIANI*,
in Celeberrima Academia OXONIENSI,
DE
ALGEBRA
Tractatus;
HISTORICUS & PRACTICUS.
Anno 1685 *Anglice editus*; *Nunc Auctus Latine*.

Cum variis
APPENDICIBUS;
Partim prius editis Anglice, Partim nunc primum editis.

Operum Mathematicorum Volumen alterum.

OXONIÆ,
E THEATRO SHELDONIANO MDCXCIII.

BIBLIOTECA DEL OBSERVATORIO DE S. FERNANDO


CAP. CIX.

Perforatio Cubi, alterum ipſi æqualem recipiens.

D. *Rupertus* Princeps Palatinus, dum in Aula Regis Angliæ *Caroli* II. verſabatur, Vir magno ingenio & ſagacitate, affirmavit aliquando, omnino fieri poſſe (& poſito pignore ſe facturum ſuſcepit) ut, æqualium Cuborum, per foramen in eorum altero factum, tranſeat alter. Quod & ipſum, audio, præſtitiſſe. Id quomodo fiat, jam ſumus oſtenſuri.

PROBLEMA.

Duorum Cuborum inter ſe æqualium, alterum ſic Excavare, ut per eum tranſeat reliquus Integer.

SOLUTIO.

Sit Integri (reliquum tranſituri) Baſis, HIKL quadratum; Cujus Latus HI = 1. Huic æqualis excavatus ABDG, intelligatur Sphæræ inſcriptus, Diametrum habenti (Cubi Diagonio æqualem) BE = √3 = 1.7320508 proxime.

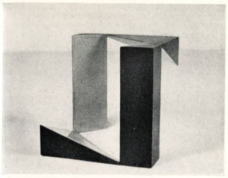


Fig. 4a

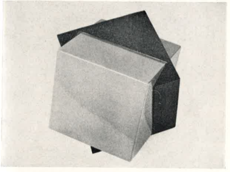
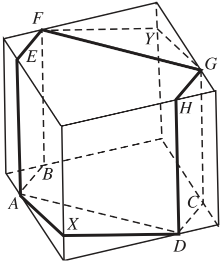
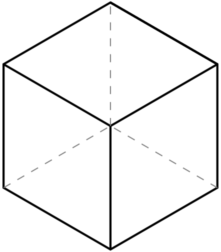


Fig. 4b

Definition of Rupert's problem





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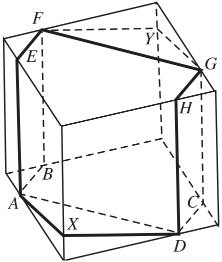
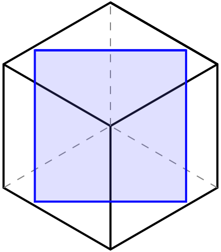
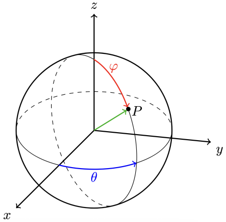
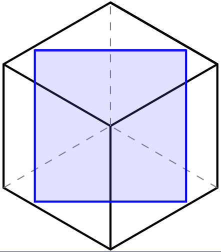




Definition of Rupert's problem

Let $M(\theta,\varphi):\mathbb{R}^3\to\mathbb{R}^2$ be an orthogonal projection map in direction $X(\theta,\varphi)\in\mathbb{R}^3$ and $R(\alpha):\mathbb{R}^2\to\mathbb{R}^2$ be the rotation map.





$$X(\theta,\varphi) := (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)^t,$$

$$M(\theta,\varphi) := \begin{pmatrix} -\sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta)\cos(\varphi) & -\sin(\theta)\cos(\varphi) & \sin(\varphi), \end{pmatrix},$$

$$R(\alpha) := \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Definition

A point-symmetric polyhedron $\mathbf{P}$ has Rupert's property, if there exist 5 parameters $\alpha,\theta_1,\theta_2\in[0,2\pi)$ and $\varphi_1,\varphi_2\in[0,\pi]$ such that

$$R(\alpha)\circ M(\theta_1,\varphi_1)\mathbf{P}\subset(M(\theta_2,\varphi_2)\mathbf{P})^\circ.$$

Brief history of Rupert's problem

- The Cube is Rupert [conjectured by Prince Rupert, proved by Wallis 1685].
- The "optimal" solution for the Cube [Nieuwland, 1816].

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Conjecture (Jerrard, Wetzel, Yuan, 2017 and Chai, Yuan, Zamfirescu, 2018)

All convex polyhedra have Rupert's property.

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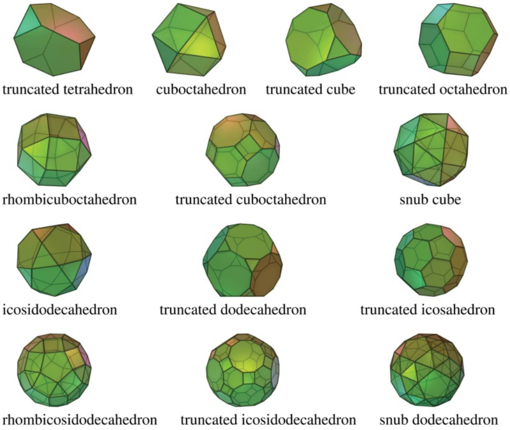
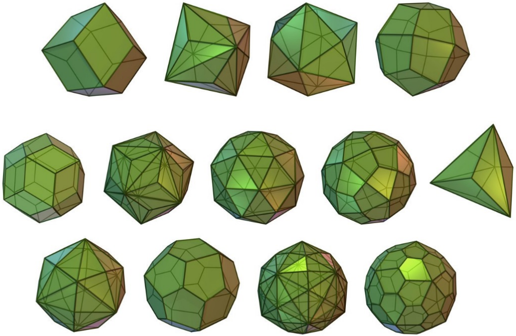
- The **Noperthedron**: a counter example to the conjecture [S., Y., 2025].

Solids

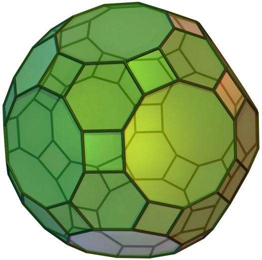

truncated tetrahedron
cuboctahedron
truncated cube
truncated octahedron
rhombicuboctahedron
truncated cuboctahedron
snub cube
icosidodecahedron
truncated dodecahedron
truncated icosahedron
rhombicosidodecahedron
truncated icosidodecahedron
snub dodecahedron

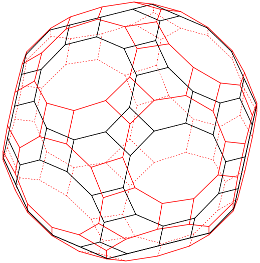


The Truncated Icosidodecahedron

Theorem (S., Y., 2021)

The Truncated Icosidodecahedron has Rupert's property.





The Truncated Icosidodecahedron

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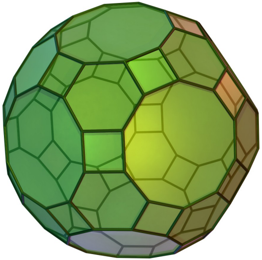
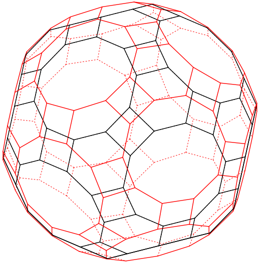
Proof.

$\alpha = 0.43584, \theta_1 = 2.77685, \theta_2 = 0.79061, \varphi_1 = 2.09416, \varphi_2 = 2.89674,$
plus some verification of linear inequalities in Maple/SageMath. □



$$R(\alpha) \circ M(\theta_1, \varphi_1)\mathbf{P} \subset (M(\theta_2, \varphi_2)\mathbf{P})^\circ$$

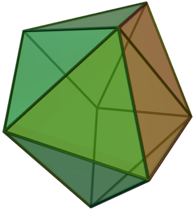
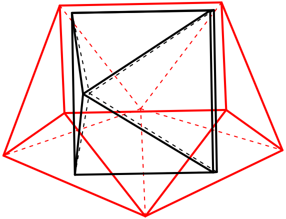
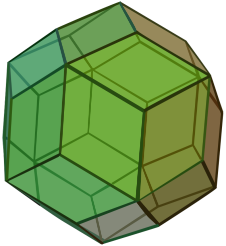
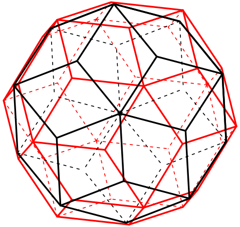


Platonic, Archimedean, Catalan and Johnson Solids

Theorem (S., Y., 2021)

In a few minutes it can be automatically proven that:

1. *All 5 Platonic solids are Rupert.*
2. *At least 10 out of 13 Archimedean solids are Rupert.*
3. *At least 9 out of 13 Catalan solids are Rupert.*
4. *At least 82 out of 92 Johnson solids are Rupert.*

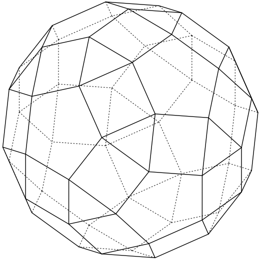





Are all convex polyhedra Rupert?

Conjecture (Jerrard, Wetzel, Yuan, 2017 and Chai, Yuan, Zamfirescu, 2018)

All convex polyhedra have Rupert's property.

- All Platonic solids are Rupert.
- 3 Archimedean solids remain open. One of them is point-symmetric.



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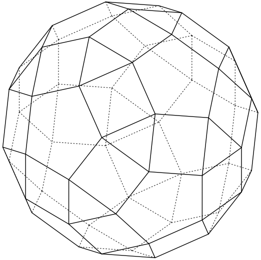

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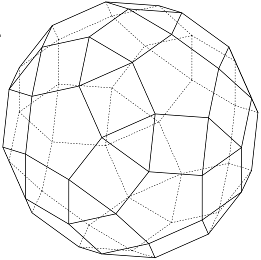
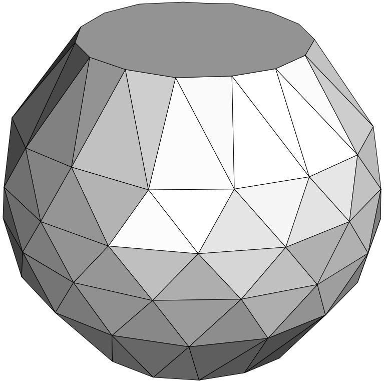
- Rhombicosidodecahedron natural candidate for disproving conjecture.
- *Rupertness*: Probability that a random projection yields a solution.
- Can estimate confidence intervals for Rupertness.
- Conclusion: RID is significantly different from other regular solids. It is likely that RID is a counter example to the conjecture.



4 years later: The Noperthedron is not Rupert

Theorem (S., Y., 2025)

*The Noperthedron, **NOP**, does not have Rupert's property.*



Idea of proof

- We need to show that

$$R(\alpha) \circ M(\theta_1, \varphi_1)\textbf{NOP} \subset (M(\theta_2, \varphi_2)\textbf{NOP})^\circ$$

has no solution. Partition the five-dimensional solution space

$$I = [0, 2\pi) \times [0, \pi] \times [0, 2\pi) \times [0, \pi] \times [-\pi, \pi)$$

into small parts and prove for each that no solution in that region exists.

- Rough idea: Show that the "middle point" of any region does not yield a solution and argue with effective continuity of the parameters that this also excludes an explicit neighborhood around that point.

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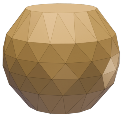
  into small parts and prove for each that no solution in that region exists.
- Rough idea: Show that the "middle point" of any region does not yield a solution and argue with effective continuity of the parameters that this also excludes an explicit neighborhood around that point.
- The **global theorem** is tailored for two generic projections of $\mathbf{P}$, when some vertex of the "smaller" projection $\mathcal{P} = R(\alpha)M(\theta_1, \varphi_1)\,\mathbf{P}$ is *strictly outside* the "larger" projection $\mathcal{Q} = M(\theta_2, \varphi_2)\,\mathbf{P}$.
- The **local theorem** can handle projections that look (almost) exactly the same, for instance if $\theta_1 \approx \theta_2$, $\varphi_1 \approx \varphi_2$ and $\alpha \approx 0$.

Noperthedron

A convex polyhedron without the Rupert property

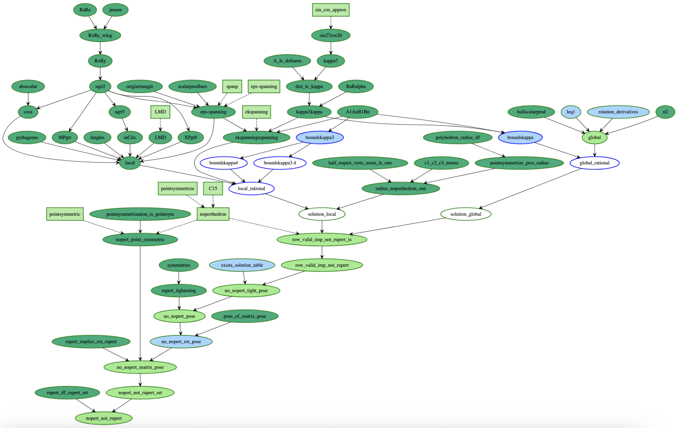
Blueprint (web) | Blueprint (pdf) | Documentation | GitHub



This is a collaborative project to write a Lean 4 proof that the Noperthedron does not have the Rupert property.

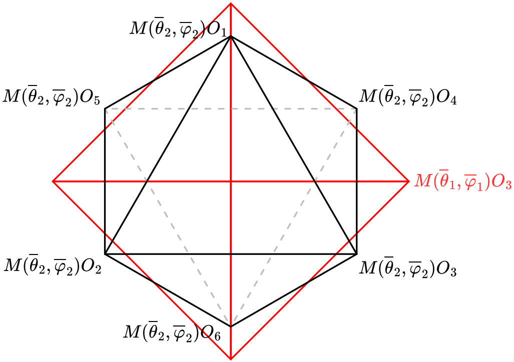
Internal links:

- Blueprint
- Blueprint as pdf
- Dependency graph
- Doc pages for this repository



Community Project formalizing the proof in LEAN by J. Reed and D. Renshaw.

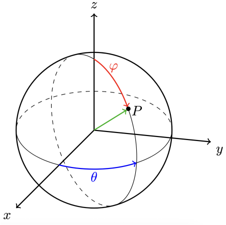
Motivation for global theorem



$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) = (0, 0, \pi/4, \tan^{-1}(\sqrt{2}), 0)$$

Starting point of the global theorem

Recall:



$$X(\theta,\varphi) := (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)^t,$$

$$M(\theta,\varphi) := \begin{pmatrix} -\sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta)\cos(\varphi) & -\sin(\theta)\cos(\varphi) & \sin(\varphi), \end{pmatrix},$$

$$R(\alpha) := \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Lemma

Let $\varepsilon > 0$ *and* $|\theta - \overline{\theta}|, |\varphi - \overline{\varphi}|, |\alpha - \overline{\alpha}| \leq \varepsilon$ *then*

- $\|X(\theta,\varphi) - X(\overline{\theta},\overline{\varphi})\| < \sqrt{2}\varepsilon,$
- $\|M(\theta,\varphi) - M(\overline{\theta},\overline{\varphi})\| < \sqrt{2}\varepsilon,$
- $\|R(\alpha)M(\theta,\varphi) - R(\overline{\alpha})M(\overline{\theta},\overline{\varphi})\| < \sqrt{5}\varepsilon.$

First version of the global theorem

Recall the definition of Rupert's property

$$R(\alpha)M(\theta_1,\varphi_1)\,\mathbf{P} \subset (M(\theta_2,\varphi_2)\,\mathbf{P})^\circ.$$

If $\Psi \in \mathbb{R}^5$ is a solution, then for any vector $w \in \mathbb{R}^2$ it holds that

$$\langle R(\alpha)M(\theta_1,\varphi_1)S, w\rangle < \max_{P\in\mathbf{P}}\langle M(\theta_2,\varphi_2)P, w\rangle.$$

Theorem (Global theorem v0.1)

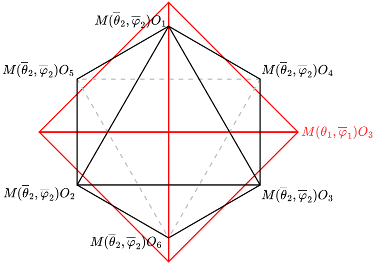
Let $\mathbf{P}$ *be convex, pointsymmetric with radius* 1. *Assume:*

$$\langle R(\overline{\alpha})M(\overline{\theta}_1,\overline{\varphi}_1)S, w\rangle > \max_{P\in\mathbf{P}}\langle M(\overline{\theta}_2,\overline{\varphi}_2)P, w\rangle + (\sqrt{2}+\sqrt{5})\varepsilon$$

for some $S \in \mathbf{P}$ *and* $w \in \mathbb{R}^2$ *with* $\|w\| = 1$, *then there cannot be a solution* $(\theta_1,\varphi_1,\theta_2,\varphi_2,\alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon]$.

Example

- Consider $\mathbf{O} = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\} \subseteq \mathbb{R}^3$ and two projection directions $(\overline{\theta}_1, \overline{\varphi}_1) = (0, 0)$ and $(\overline{\theta}_2, \overline{\varphi}_2) = (\pi/4, \tan^{-1}(\sqrt{2}))$. Set $\overline{\alpha} = 0$.
- Goal: show this is no solution to Rupert's problem and there exists $\varepsilon > 0$ such that there is also no solution $(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha)$ with $|\overline{\theta}_i - \theta_i|, |\overline{\varphi}_i - \varphi_i|, |\alpha| \leq \varepsilon$.



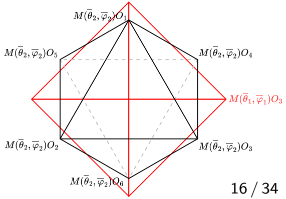
Example continued

Following the global theorem we first compute:

$$M(\overline{\theta}_1,\overline{\varphi}_1)=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix} \quad \text{and} \quad M(\overline{\theta}_2,\overline{\varphi}_2)=\begin{pmatrix}-\sqrt{2}/2 & \sqrt{2}/2 & 0\\ -\sqrt{6}/6 & -\sqrt{6}/6 & \sqrt{6}/3\end{pmatrix}.$$

We choose $S=O_3=(0,1,0)$ and $w=(1,0)$, thus $\langle R(\overline{\alpha})M(\overline{\theta}_1,\overline{\varphi}_1)S,w\rangle=1$.

$$\langle M(\overline{\theta}_2,\overline{\varphi}_2)P,w\rangle=\begin{cases}0 & \text{for } P=M(\overline{\theta}_2,\overline{\varphi}_2)O_i \ \text{ with } \ i=1,6,\\ \sqrt{2}/2 & \text{for } P=M(\overline{\theta}_2,\overline{\varphi}_2)O_i \ \text{ with } \ i=3,4,\\ -\sqrt{2}/2 & \text{for } P=M(\overline{\theta}_2,\overline{\varphi}_2)O_i \ \text{ with } \ i=2,5.\end{cases}$$



Example continued

Following the global theorem we first compute:

$$M(\overline{\theta}_1,\overline{\varphi}_1)=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix} \quad \text{and} \quad M(\overline{\theta}_2,\overline{\varphi}_2)=\begin{pmatrix}-\sqrt{2}/2 & \sqrt{2}/2 & 0\\ -\sqrt{6}/6 & -\sqrt{6}/6 & \sqrt{6}/3\end{pmatrix}.$$

We choose $S=O_3=(0,1,0)$ and $w=(1,0)$, thus $\langle R(\overline{\alpha})M(\overline{\theta}_1,\overline{\varphi}_1)S,w\rangle=1$.

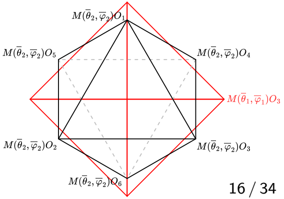
$$\langle M(\overline{\theta}_2,\overline{\varphi}_2)P,w\rangle=\begin{cases}0 & \text{for } P=M(\overline{\theta}_2,\overline{\varphi}_2)O_i \text{ with } i=1,6,\\ \sqrt{2}/2 & \text{for } P=M(\overline{\theta}_2,\overline{\varphi}_2)O_i \text{ with } i=3,4,\\ -\sqrt{2}/2 & \text{for } P=M(\overline{\theta}_2,\overline{\varphi}_2)O_i \text{ with } i=2,5.\end{cases}$$

Thus if $\varepsilon>0$ is chosen so that

$$1>\sqrt{2}/2+(\sqrt{2}+\sqrt{5})\varepsilon, \qquad \text{e.g., } \varepsilon=0.08$$

there is no solution in

$$(0,0,\pi/4,\tan^{-1}(\sqrt{2}),0)\pm\varepsilon\subseteq\mathbb{R}^5.$$



Actual Global theorem

Theorem (Global Theorem v1.0)

Let $\mathbf{P}$ *be a pointsymmetric convex polyhedron with radius* 1 *and let* $S \in \mathbf{P}$. *Let* $w \in \mathbb{R}^2$ *be a unit vector and denote* $\overline{M_1} := M(\overline{\theta}_1, \overline{\varphi}_1)$, $\overline{M_2} := M(\overline{\theta}_2, \overline{\varphi}_2)$ *as well as* $\overline{M_1}^{\theta} := M^{\theta}(\overline{\theta}_1, \overline{\varphi}_1)$, $\overline{M_1}^{\varphi} := M^{\varphi}(\overline{\theta}_1, \overline{\varphi}_1)$ *and analogously for* $\overline{M_2}^{\theta}, \overline{M_2}^{\varphi}$. *Finally set*

$$G := \langle R(\overline{\alpha})\overline{M_1}S, w\rangle - \varepsilon \cdot \left(|\langle R'(\overline{\alpha})\overline{M_1}S, w\rangle| + |\langle R(\overline{\alpha})\overline{M_1}^{\theta}S, w\rangle| + |\langle R(\overline{\alpha})\overline{M_1}^{\varphi}S, w\rangle|\right) - 9\varepsilon^2/2,$$

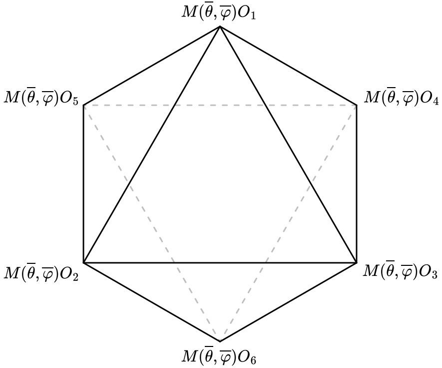
$$H_P := \langle \overline{M_2}P, w\rangle + \varepsilon \cdot \left(|\langle \overline{M_2}^{\theta}P, w\rangle| + |\langle \overline{M_2}^{\varphi}P, w\rangle|\right) + 2\varepsilon^2, \qquad \text{for } P \in \mathbf{P}.$$

If $G > \max_{P \in \mathbf{P}} H_P$ *then there does not exist a solution to Rupert's condition with*

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon] \subseteq \mathbb{R}^5.$$

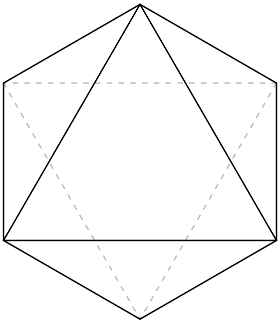
$\varepsilon = 0.08$ from the Example can be replaced by $\varepsilon = 0.164$

Motivation for local theorem: $\theta_1 = \theta_2$, $\varphi_1 = \varphi_2$, $\alpha = 0$



$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) = (\pi/4, \tan^{-1}(\sqrt{2}), \pi/4, \tan^{-1}(\sqrt{2}), 0)$$

Motivation for local theorem: $\theta_1 = \theta_2$, $\varphi_1 = \varphi_2$, $\alpha = 0$



$$A := M(\theta,\varphi)O_1,$$
$$B := M(\theta,\varphi)O_2,$$
$$C := M(\theta,\varphi)O_3$$

Fact:

$$\|A\|^2 + \|B\|^2 + \|C\|^2 = 2$$

$\implies$ no local solution from this direction

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) = (\pi/4, \tan^{-1}(\sqrt{2}), \pi/4, \tan^{-1}(\sqrt{2}), 0)$$

Conditions for the three-point method

- P_1, P_2, P_3 are all in front of the projection:

$$\langle X(\overline{\theta}, \overline{\varphi}), P_i \rangle > 0.$$

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- The projected points $M(\overline{\theta}, \overline{\varphi}) \cdot \{P_1, P_2, P_3\}$ are not to close to the origin:

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- The points P_1, P_2, P_3 are locally maximally distant.

Conditions for the three-point method

- P_1, P_2, P_3 are all in front of the projection:

$$\langle X(\overline{\theta}, \overline{\varphi}), P_i \rangle > \sqrt{2}\varepsilon.$$

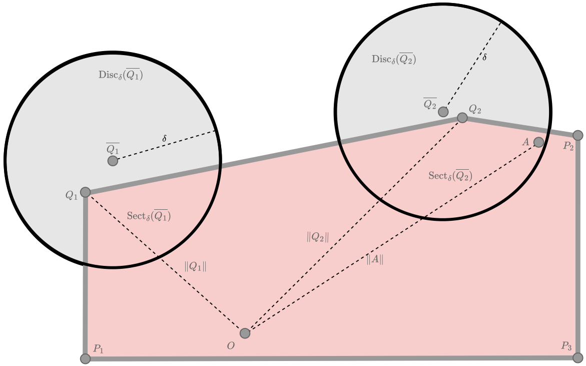
- The origin is inside the projected triangle $M(\overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon) \cdot \{P_1, P_2, P_3\}$.
  $:\Leftrightarrow P_1, P_2, P_3$ are ε-spanning

- The projected points $M(\overline{\theta}, \overline{\varphi}) \cdot \{P_1, P_2, P_3\}$ are not to close to the origin:

$$\|M(\overline{\theta}, \overline{\varphi}) P_i\| > r + \sqrt{2}\varepsilon.$$

- The points P_1, P_2, P_3 are ε-locally maximally distant.

Locally maximally distant points



Local Theorem

Theorem (Local Theorem v0.1)

Let $\mathbf{P}$ *be a convex, pointsymmetric polyhedron with radius 1 and* $P_1, P_2, P_3 \in \mathbf{P}$. *Set* $\overline{X} := X(\overline{\theta}, \overline{\varphi})$, $\overline{M} := M(\overline{\theta}, \overline{\varphi})$. *Assume that for all* $i = 1, 2, 3$:

- $\langle \overline{X}, P_i \rangle > \sqrt{2}\varepsilon$,
- P_1, P_2, P_3 *are* ε*-spanning for* $(\overline{\theta}, \overline{\varphi})$,
- *the points* P_1, P_2, P_3 *are* ε*-LMD.*

Then there exists no solution to Rupert's problem with

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon, \overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon, \pm\varepsilon].$$

ε-spanning points

Lemma

Let P_1, P_2, P_3 *with* $\|P_1\|, \|P_2\|, \|P_3\| \leq 1$ *be* ε*-spanning for* $(\overline{\theta}, \overline{\varphi})$ *and let* $\theta, \varphi \in \mathbb{R}$ *such that* $|\theta - \overline{\theta}|, |\varphi - \overline{\varphi}| \leq \varepsilon$*. Assume that* $\langle X(\theta, \varphi), P_i \rangle > 0$ *for* $i = 1, 2, 3$*. Then*

$$X(\theta, \varphi) \in \operatorname{span}^{+}(P_1, P_2, P_3).$$

Some more lemmas

Lemma (Pythagoras)

For any $P \in \mathbb{R}^3$ *one has* $\left\|M(\theta,\varphi)P\right\|^2 = \|P\|^2 - \langle X(\theta,\varphi), P\rangle^2$.

Lemma (Trinity)

Let $P_1, P_2, P_3, Y, Z \in \mathbb{R}^3$ *with* $\|Y\| = \|Z\|$ *and* $Y, Z \in \operatorname{span}^+(P_1, P_2, P_3)$. *Then at least one of the following inequalities does not hold:*

$$\langle P_1, Y\rangle > \langle P_1, Z\rangle,$$
$$\langle P_2, Y\rangle > \langle P_2, Z\rangle,$$
$$\langle P_3, Y\rangle > \langle P_3, Z\rangle.$$

Proof sketch of the Local Theorem

Proof sketch.

1. Assume $(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon, \overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon, \pm\varepsilon]$, let

$$M_1 = M(\theta_1, \varphi_1), M_2 = M(\theta_2, \varphi_2), X_1 = X(\theta_1, \varphi_1), X_2 = X(\theta_2, \varphi_2).$$

Proof sketch of the Local Theorem

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1. Assume $(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon, \overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon, \pm\varepsilon]$, let

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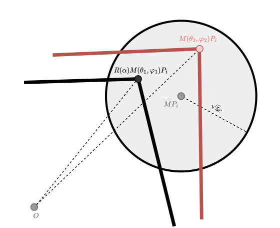
2. Rupert's property + ε-LMD $\Rightarrow \|M_2 P_i\| > \|M_1 P_i\|$ for $i = 1, 2, 3$.



Proof sketch of the Local Theorem

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$$M_1 = M(\theta_1, \varphi_1), M_2 = M(\theta_2, \varphi_2), X_1 = X(\theta_1, \varphi_1), X_2 = X(\theta_2, \varphi_2).$$
2. Rupert's property + ε-LMD $\Rightarrow \|M_2 P_i\| > \|M_1 P_i\|$ for $i = 1, 2, 3$.
3. $\langle \overline{X}, P_i \rangle > \sqrt{2}\varepsilon \Rightarrow \langle X_1, P_i \rangle, \langle X_2, P_i \rangle > 0$.

Proof sketch of the Local Theorem

Proof sketch.

1. Assume $(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon, \overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon, \pm\varepsilon]$, let
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4. (2), (3) and (Pythagoras) $\Rightarrow \langle X_1, P_i \rangle > \langle X_2, P_i \rangle$

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4. (2), (3) and (Pythagoras) $\Rightarrow \langle X_1, P_i \rangle > \langle X_2, P_i \rangle$
5. P_1, P_2, P_3 are ε-spanning (+ Lemma)$\Rightarrow X_1, X_2 \in \mathrm{span}^+(P_1, P_2, P_3)$

Proof sketch of the Local Theorem

Proof sketch.

1. Assume $(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon, \overline{\theta} \pm \varepsilon, \overline{\varphi} \pm \varepsilon, \pm\varepsilon]$, let
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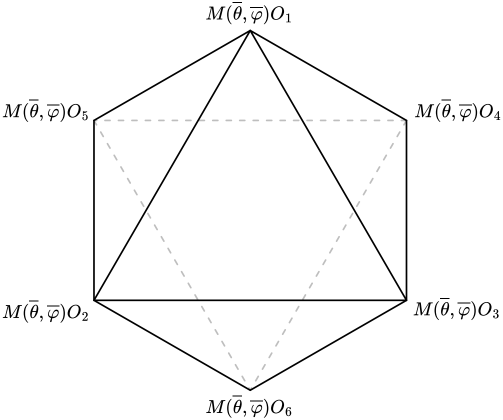
5. P_1, P_2, P_3 are ε-spanning (+ Lemma)$\Rightarrow X_1, X_2 \in \mathrm{span}^+(P_1, P_2, P_3)$
6. (4)+(5) + Trinity lemma $\Rightarrow$ contradiction.

□

Example continued



Theorem (Local Theorem v1.0)

Let $\mathbf{P}$ *be a polyhedron with radius* 1 *and* $P_1, P_2, P_3, Q_1, Q_2, Q_3 \in \mathbf{P}$. *Assume that* P_1, P_2, P_3 *and* Q_1, Q_2, Q_3 *are congruent. Let* $\varepsilon > 0$ *and* $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{R}$, *then set* $\overline{X_1} := X(\overline{\theta}_1, \overline{\varphi}_1), \overline{X_2} := X(\overline{\theta}_2, \overline{\varphi}_2)$ *as well as* $\overline{M_1} := M(\overline{\theta}_1, \overline{\varphi}_1), \overline{M_2} := M(\overline{\theta}_2, \overline{\varphi}_2)$. *Assume that there exist* $\sigma_P, \sigma_Q \in \{0, 1\}$ *such that*

$$(-1)^{\sigma_P} \langle \overline{X_1}, P_i \rangle > \sqrt{2}\varepsilon \quad \text{and} \quad (-1)^{\sigma_Q} \langle \overline{X_2}, Q_i \rangle > \sqrt{2}\varepsilon, \tag{A_ε}$$

for all $i = 1, 2, 3$. *Moreover, assume that* P_1, P_2, P_3 *are* ε*-spanning for* $(\overline{\theta}_1, \overline{\varphi}_1)$ *and that* Q_1, Q_2, Q_3 *are* ε*-spanning for* $(\overline{\theta}_2, \overline{\varphi}_2)$. *Finally, assume that for all* $i = 1, 2, 3$ *and any* $Q_j \in \mathbf{P} \setminus Q_i$ *it holds that*

$$\frac{\langle \overline{M_2} Q_i, \overline{M_2}(Q_i - Q_j) \rangle - 2\varepsilon \|Q_i - Q_j\| \cdot (\sqrt{2} + \varepsilon)}{(\|\overline{M_2} Q_i\| + \sqrt{2}\varepsilon) \cdot (\|\overline{M_2}(Q_i - Q_j)\| + 2\sqrt{2}\varepsilon)} > \frac{\sqrt{5}\varepsilon + \delta}{r}, \tag{B_ε}$$

for some $r > 0$ *such that* $\min_{i=1,2,3} \|\overline{M_2} Q_i\| > r + \sqrt{2}\varepsilon$ *and for some* $\delta \in \mathbb{R}$ *with*

$$\delta \geq \max_{i=1,2,3} \left\| R(\overline{\alpha}) \overline{M_1} P_i - \overline{M_2} Q_i \right\| / 2.$$

Then there exists no solution to Rupert's problem $R(\alpha) M(\theta_1, \varphi_1) \mathbf{P} \subset M(\theta_2, \varphi_2) \mathbf{P}^\circ$ *with*

$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon] \subseteq \mathbb{R}^5.$$

Motivation for rational approximation

```
R 4.5.1 · ~/
> 1 + 10^(-16) > 1
[1] FALSE
>
```

```
SageMath version 10.0, Release Date: 2023-05-20
Using Python 3.11.1. Type "help()" for help.

sage: 1 + 10^(-16) > 1
True
sage:
```

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Using Python 3.11.1. Type "help()" for help.

sage: 1 + 10^(-16) > 1
True
sage:
```

imprecise but fast

exact but slow

Idea of rational approximation $\kappa := 10^{-10}$

$$\sin_{\mathbb{Q}}(x) := x - \frac{x^3}{3} + \frac{x^5}{5!} \mp \cdots + \frac{x^{25}}{25!},$$
$$\cos_{\mathbb{Q}}(x) := 1 - \frac{x^2}{2} + \frac{x^4}{4!} \mp \cdots + \frac{x^{24}}{24!}.$$

By replacing $\sin, \cos$ with $\sin_{\mathbb{Q}}, \cos_{\mathbb{Q}}$ define the functions

$$R_{\mathbb{Q}}(\alpha), R'_{\mathbb{Q}}(\alpha), X_{\mathbb{Q}}(\theta,\varphi), M_{\mathbb{Q}}(\theta,\varphi), M^{\theta}_{\mathbb{Q}}(\theta,\varphi), M^{\varphi}_{\mathbb{Q}}(\theta,\varphi).$$

Idea of rational approximation $\kappa := 10^{-10}$

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Lemma

Let $\alpha, \theta, \varphi \in [-4, 4]$. Then it holds that

$$\|R(\alpha) - R_{\mathbb{Q}}(\alpha)\|, \|R'(\alpha) - R'_{\mathbb{Q}}(\alpha)\|, \|X(\theta,\varphi) - X_{\mathbb{Q}}(\theta,\varphi)\|, \|M(\theta,\varphi) - M_{\mathbb{Q}}(\theta,\varphi)\| \le \kappa.$$

Moreover,

$$\|R_{\mathbb{Q}}(\alpha)\|, \|R'_{\mathbb{Q}}(\alpha)\|, \|X_{\mathbb{Q}}(\theta,\varphi)\|, \|M_{\mathbb{Q}}(\theta,\varphi)\| \le 1 + \kappa.$$

Rational global theorem

Theorem (Rational Global Theorem)

Let $\mathbf{P}$ *be a pointsymmetric convex polyhedron with radius* $\rho = 1$ *and* $\widetilde{\mathbf{P}}$ *a* *κ-rational approximation. Let* $\widetilde{S} \in \widetilde{\mathbf{P}}$*. Further let* $\varepsilon > 0$ *and* $\overline{\theta}_1, \overline{\varphi}_1, \overline{\theta}_2, \overline{\varphi}_2, \overline{\alpha} \in \mathbb{Q} \cap [-4, 4]$*. Let* $w \in \mathbb{Q}^2$ *be a unit vector. Denote* $\overline{M_1} := M_{\mathbb{Q}}(\overline{\theta}_1, \overline{\varphi}_1)$*,* $\overline{M_2} := M_{\mathbb{Q}}(\overline{\theta}_2, \overline{\varphi}_2)$ *as well as* $\overline{M_1}^{\theta} := M_{\mathbb{Q}}^{\theta}(\overline{\theta}_1, \overline{\varphi}_1)$*,* $\overline{M_1}^{\varphi} := M_{\mathbb{Q}}^{\varphi}(\overline{\theta}_1, \overline{\varphi}_1)$ *and analogously for* $\overline{M_2}^{\theta}, \overline{M_2}^{\varphi}$*. Finally set*

$$G^{\mathbb{Q}} := \langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}\widetilde{S}, w\rangle - \varepsilon \cdot \left(|\langle R'_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}\widetilde{S}, w\rangle| + |\langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}^{\theta}\widetilde{S}, w\rangle| + |\langle R_{\mathbb{Q}}(\overline{\alpha})\overline{M_1}^{\varphi}\widetilde{S}, w\rangle|\right) - 9\varepsilon^2/2 - 4\kappa(1 + 3\varepsilon),$$

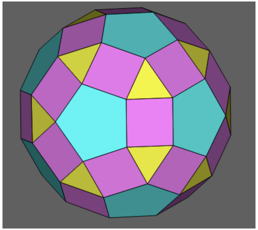
$$H_P^{\mathbb{Q}} := \langle \overline{M_2}P, w\rangle + \varepsilon \cdot \left(|\langle \overline{M_2}^{\theta}P, w\rangle| + |\langle \overline{M_2}^{\varphi}P, w\rangle|\right) + 2\varepsilon^2 + 3\kappa(1 + 2\varepsilon).$$

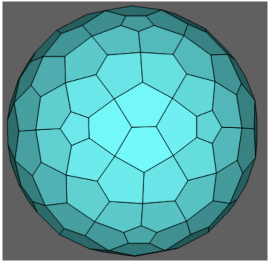
If $G^{\mathbb{Q}} > \max_{P \in \widetilde{\mathbf{P}}} H_P^{\mathbb{Q}}$ *then there does not exist a solution to Rupert's condition with*

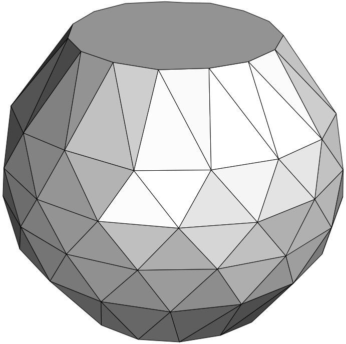
$$(\theta_1, \varphi_1, \theta_2, \varphi_2, \alpha) \in [\overline{\theta}_1 \pm \varepsilon, \overline{\varphi}_1 \pm \varepsilon, \overline{\theta}_2 \pm \varepsilon, \overline{\varphi}_2 \pm \varepsilon, \overline{\alpha} \pm \varepsilon].$$

Wishlist for a solid

1. Not Rupert
2. many symmetries
3. no mirrorsymmetry because of (A_ε) and (B_ε)
4. local theorem always applicable
5. pointsymmetry
6. not too many vertices







Definition of the Noperthedron

$$\mathcal{C}_{30} := \left\{ (-1)^{\ell} R_z\left(\frac{2\pi k}{15}\right) : k = 0, \dots, 14; \ell = 0, 1 \right\}.$$

$$C_1 := \frac{1}{259375205}\begin{pmatrix}152024884\\0\\210152163\end{pmatrix}, \quad C_2 := \frac{1}{10^{10}}\begin{pmatrix}6632738028\\6106948881\\3980949609\end{pmatrix}, \quad C_3 := \frac{1}{10^{10}}\begin{pmatrix}8193990033\\5298215096\\1230614493\end{pmatrix}.$$

Note: $\|C_1\| = 1$ and $\frac{98}{100} < \|C_i\| < \frac{99}{100}$ for $i = 2, 3$.

Definition

Define the set of points **NOP** $\subseteq \mathbb{R}^3$ by the action of $\mathcal{C}_{30}$ on C_1, C_2, C_3:

$$\mathbf{NOP} := \mathcal{C}_{30} \cdot C_1 \cup \mathcal{C}_{30} \cdot C_2 \cup \mathcal{C}_{30} \cdot C_3.$$

The *Noperthedron* has 90 vertices. **NOP** is pointsymmetric since $-\mathrm{Id} \in \mathcal{C}_{30}$.
Symmetries, e.g.: $M(\theta + 2\pi/15, \varphi) \cdot \mathbf{NOP} = M(\theta, \varphi) \cdot \mathbf{NOP} \Rightarrow \theta_1, \theta_2 \in [0, 2\pi/15)$.

Certificate of computer proof

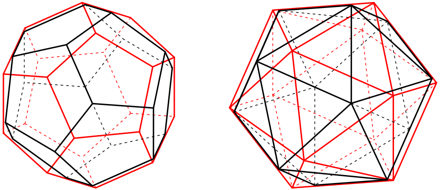
ID	T	#C	ID1C	sp	$\overline{\theta}_{1,\min}$	$\overline{\theta}_{1,\max}$	$\overline{\varphi}_{1,\min}$	$\overline{\varphi}_{1,\max}$	$\overline{\theta}_{2,\min}$	$\overline{\theta}_{2,\max}$	$\overline{\varphi}_{2,\min}$	$\overline{\varphi}_{2,\max}$	$\overline{\alpha}_{\min}$	$\overline{\alpha}_{\max}$	P_1	P_2	P_3	Q_1	Q_2	Q_3	r	σ_Q	w_x	w_y	w_d	S
0	3	4	1	1	0	64..00	0	48..00	0	64..00	0	24..00	-24..00	24..00												
1	3	30	5	2	0	16..00	0	48..00	0	64..00	0	24..00	-24..00	24..00												
2	3	30	46..67	2	16..00	32..00	0	48..00	0	64..00	0	24..00	-24..00	24..00												
3	3	30	94..77	2	32..00	48..00	0	48..00	0	64..00	0	24..00	-24..00	24..00												
4	3	30	14..51	2	48..00	64..00	0	48..00	0	64..00	0	24..00	-24..00	24..00												
5	3	4	35	3	0	16..00	0	16..00	0	64..00	0	24..00	-24..00	24..00												
6	3	4	70..31	3	0	16..00	16..00	32..00	0	64..00	0	24..00	-24..00	24..00												
7	3	4	10..67	3	0	16..00	32..00	48..00	0	64..00	0	24..00	-24..00	24..00												
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
⋮ 29	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
91	1				0	80..00	0	80..00	80..00	16..00	80..60	16..20	-23..40	-22..80									53..73	15..64	16..45	39
92	1				0	80..00	80..00	16..00	0	80..00	0	80..60	-24..00	-23..40									98..92	35..15	10..33	37
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
245	2				0	20..00	0	20..00	0	20..00	0	20..40	-22..20	-22..80	30	31	38	79	80	87	-69	1				
246	1				0	20..00	0	20..00	0	20..00	20..40	40..80	-23..60	-22..20									71..05	19..88	20..37	39
247	1				0	20..00	0	20..00	0	20..00	20..40	40..80	-22..20	-22..80									71..05	19..88	20..37	39
248	2				0	20..00	0	20..00	20..00	40..00	0	20..40	-23..60	-22..20	30	31	38	79	80	87	-69	1				
249	2				0	20..00	0	20..00	20..00	40..00	0	20..40	-22..20	-22..80	30	31	38	79	80	87	-69	1				
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
18..44	1				48..00	64..00	46..00	48..00	48..00	64..00	22..80	24..00	22..80	24..00									33..40	-14..51	14..49	78

Computer proof statistics

- $\approx$ 18.000.000 global theorem applications
- $\approx$ 600.000 local theorem applications
- $\approx$ 3Gb uncompressed certificate ($\approx$ 150Mb compressed)
- $\approx$ 10h for creation of table (using floating points in R)
- $\approx$ 30h for verification in SageMath

Conclusion and open question

- What about the remaining 3 Archimedean solids? In particular the Rhombicosidodecahedron?
- Is there a way to prove that a solid does not have Rupert's property without a huge case distinction? In particular, are there other ways to disprove the existence of local solutions?
- How to prove the conjectured optimal solutions?
- What is the link to duality?



Bonus: Nieuwland's number

Fact (Nieuwland, 1816)

It is possible to cut a hole inside the **unit cube** such that a **cube with side length less than** $3\sqrt{2}/4 \approx 1.0606$ can pass through this hole.

- The largest number $\nu \in \mathbb{R}$ such that $\nu\mathbf{P}$ passes through some hole inside $\mathbf{P}$ is called *Nieuwland number of* $\mathbf{P}$.
- For all solids it holds that $\nu \geq 1$.
- $\mathbf{P}$ is Rupert $\iff$ $\nu(\mathbf{P}) > 1$.
- ν of the Cube is exactly $3\sqrt{2}/4$ [Nieuwland, 1816].

Bonus: Sufficient condition of LMD

Lemma

Let $\mathcal{P}$ be a convex polygon and $Q \in \mathcal{P}$. Let $\overline{Q} \in \mathbb{R}^2$ with $\|Q - \overline{Q}\| < \delta$ for some $\delta > 0$. Assume that for some $r > 0$ such that $\|Q\| > r$ it holds that

$$\frac{\langle Q, Q - P_j \rangle}{\|Q\| \|Q - P_j\|} \geq \frac{\delta}{r},$$

for all vertices $P_j \in \mathcal{P} \setminus Q$. Then $Q \in \mathcal{P}$ is δ-LMD$(\overline{Q})$.

Proof sketch.

Assume $A \in \mathrm{Sect}_\delta(\overline{Q}) = \mathrm{Disc}_\delta(\overline{Q}) \cap \mathcal{P}^\circ$, use $\cos(\angle(O, Q, P_j)) = \frac{\langle Q, Q - P_j \rangle}{\|Q\| \|Q - P_j\|} \geq \frac{\delta}{r}$ to conclude that $\cos(\angle(O, Q, A)) \geq \delta / r$. Therefore,

$$\|A\|^2 - \|Q\|^2 = \|Q - A\| \cdot (\|Q - A\| - 2\|Q\| \cos(\angle(O, Q, A))) < 0.$$

□

Bonus: Proof of main lemma for global theorem

$$R_x(\alpha) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, R_y(\alpha) := \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}, R_z(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$X(\theta,\varphi)^t = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \cdot R_y(\varphi) \cdot R_z(-\theta), \quad M(\theta,\varphi) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \cdot R_y(\varphi) \cdot R_z(-\theta)$$

$$R(\alpha)M(\theta,\varphi) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \cdot R_z(\alpha) \cdot R_y(\varphi) \cdot R_z(-\theta).$$

Lemma

For any $\alpha, \beta \in \mathbb{R}$ *one has* $\| R_x(\alpha) R_y(\beta) - \mathrm{Id} \| \leq \sqrt{\alpha^2 + \beta^2}$.

Sketch of proof: Write $R_x(\alpha)R_y(\beta) = UR_x(\Phi)U^{-1}$ and take trace to obtain:

$$\cos(\alpha) + \cos(\beta) + \cos(\alpha)\cos(\beta) = 1 + 2\cos(\Phi).$$

Jensen on $f(t) = \cos(\sqrt{t})$ shows LHS $\geq 1 + 2\cos(\sqrt{\alpha^2+\beta^2})$, thus $|\Phi| \leq \sqrt{\alpha^2+\beta^2}$.