# Algebraicity of solutions of DDEs with one catalytic variable[1]

Arbeitsgemeinschaft Diskrete Mathematik

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[1] Joint work with Hadrien Notarantonio. Accepted as a talk at FPSAC23.

# Motivating examples

- Let $F(t,u) = \sum_{n,k\geq 0} a_{n,k} t^n u^k$ be the generating function of walks in $\mathbb{N}^2$ which have $n$ steps in $\{\nearrow, \searrow\}$ and end at level (height) $k$. One finds:

$$F(t,u) = 1 + tuF(t,u) + t\frac{F(t,u) - F(t,0)}{u}.$$

It follows that: $F(t,0) = \dfrac{1-\sqrt{1-4t^2}}{2t^2}$. In particular: $F(t,0) \in \overline{\mathbb{Q}(t)}$.

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- Modelling [Bonichon, Bousquet-Mélou, Dorbec, Pennarun, 2006] special Eulerian planar orientations gives rise to:

$$\begin{cases} F_1(t,u) = 1 + t\cdot\left(u + 2uF_1(t,u)^2 + 2uF_2(t,1) + u\frac{F_1(t,u)-uF_1(t,1)}{u-1}\right), \\ F_2(t,u) = t\cdot\left(2uF_1(t,u)F_2(t,u) + uF_1(t,u) + uF_2(t,1) + u\frac{F_2(t,u)-uF_2(t,1)}{u-1}\right). \end{cases}$$

Again: $G = F_1(t,1)$ and $F_2(t,1)$ are algebraic functions, for example:

$$64t^3G^3 + 2t\left(24t^2 - 36t + 1\right)G^2 - \left(15t^3 - 9t^2 - 19t + 1\right)G + t^3 + 27t^2 - 19t + 1 = 0.$$

# Discrete Differential Equations (DDEs) with one catalytic variable

- The divided difference operator (discrete derivative):

$$\Delta_a : \mathbb{Q}[u][[t]] \to \mathbb{Q}[u][[t]],$$
$$F(t,u) \mapsto \frac{F(t,u) - F(t,a)}{u-a}.$$

- $\Delta_a^j$ is the $j$-th iteration of $\Delta_a$. Explicitly:

$$\Delta_a^{j+1} F(t,u) = \frac{F(t,u) - F(t,a) - (u-a)\partial_u F(t,a) - \cdots - \frac{(u-a)^j}{j!}\partial_u^j F(t,a)}{(u-a)^{j+1}}$$

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- For polynomials $f(u) \in \mathbb{Q}[u]$ and $Q \in \mathbb{Q}[x, y_1 \ldots, y_k, t, u]$ consider the equation

$$F(t,u) = f(u) + t \cdot Q(F(t,u), \Delta_a F(t,u), \ldots, \Delta_a^k F(t,u), t, u), \qquad \textbf{(DDE)}$$

where $a \in \mathbb{Q}$ (usually 0 or 1) and $k \in \mathbb{N}$ (the order of the **DDE**).

# Some history

- In 1960s, Tutte reduced many combinatorial problems to studying **DDEs**.
  E.g.: [Tutte, 1962] and [Brown, Tutte 1964]
- In 1986 Popescu proved the "General Néron desingularization" [Popescu, 1986].
- [Banderier, Flajolet 2002]: Universal "Kernel method" for linear **DDEs**.
- In 2006: The unique solution of any **DDE** is an algebraic function and effective method to compute the minimal polynomial [Bousquet-Mélou, Jehanne, 2006].
- In 2015 [Hauser, Rond] organize a conference on "Artin Approximation":
  Popescu's Theorem also implies algebraicity of **DDEs** (but in a non-effective way).
- [Buchacher, Kauers, 2020]: Linear systems of **DDEs** have algebraic solutions
  [Asinowski, Bacher, Banderier, Gittenberger, 2020] (effective proof).
- [Bostan, Chyzak, Notarantonio, Safey El Din, 2022]: Fast algorithms for order 1.
- **New:** [Notarantonio, Y., 2022]: Effective proof that systems of **DDEs** have algebraic solutions.

# Main result for scalar equations

**Theorem (Bousquet-Mélou, Jehanne, 2006)**

*Let $k \geq 1, a \in \mathbb{Q}$ and $Q \in \mathbb{Q}[x, y_1, \dots, y_k, t, u]$, $f(u) \in \mathbb{Q}[u]$. There exists a unique solution $F(t, u) \in \mathbb{Q}[u][[t]]$ of the functional equation*

$$F(t, u) = f(u) + t \cdot Q(F(t, u), \Delta_a F(t, u), \dots, \Delta_a^k F(t, u), t, u),$$

*and $F(t, u)$ is algebraic over $\mathbb{Q}(t, u)$. Moreover, there exists an algorithm for computing the minimal polynomial of $F(t, u)$.*

# Sketch of proof (generic case)

- Let $E$ be the polynomial in $x, z_0, \dots, z_{k-1}, t, u$ that is induced by **DDE**:

$$E(u) := E(\underbrace{F(t,u)}_{x}, \underbrace{F(t,0)}_{z_0}, \dots, \underbrace{\partial_u^{k-1} F(t,0)}_{z_{k-1}}, t, u) = 0.$$

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- Take the derivative of $E(u) = 0$ with respect to $u$:

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- Any solution $u = U(t)$ of $\partial_x E(u) = 0$ also implies $\partial_u E(u) = 0$.
- Obtain 3 equations $(E(u) = 0, \partial_x E(u) = 0, \partial_u E(u) = 0)$ for $k + 1 + 1$ variables.

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- If $\partial_x E(u) = 0$ has $k$ distinct solutions $U_1, \ldots, U_k$, we can consider:

$$\mathcal{S}_{\text{dup}} := \begin{cases} E(u_i) = 0, \\ \partial_x E(u_i) = 0, \\ \partial_u E(u_i) = 0. \end{cases} \quad \text{for } i = 1, \ldots, k. \qquad \begin{cases} F(t, U_i) \leftrightarrow x_i, \\ \partial_u^i F(t,0) \leftrightarrow z_i, \\ U_i \leftrightarrow u_i. \end{cases}$$

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- We find: $3k$ equations and $3k$ variables. Hope: the system $\mathcal{S}_{\text{dup}}$ is 0-dimensional.
- In that case can use elimination algorithms and find the annihilating polynomial.

# Sketch of proof

- Two issues:
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  1. $\partial_x E(u) = 0$ does not always have $k$ distinct solutions.
  2. Is $\mathcal{S}_{\text{dup}}$ really 0-dimensional?
- Given the **DDE**

$$F(t,u) = f(u) + t \cdot Q(F(t,u), \Delta_a F(t,u), \ldots, \Delta_a^k F(t,u), t, u),$$

consider the perturbed $\mathbf{DDE}_\epsilon$:

$$G(t,u,\epsilon) = f(u) + t^2 \cdot Q(G(t,u,\epsilon), \Delta_a G(t,u,\epsilon), \ldots, \Delta_a^k G(t,u,\epsilon), t^2, u) + t\epsilon^k \Delta^k G(t,u,\epsilon).$$

- $G(t,u,\epsilon)$ algebraic over $\mathbb{Q}(t,u,\epsilon) \Rightarrow F(t,u)$ algebraic over $\mathbb{Q}(t,u)$.

**Lemma 1**

Let $E_G$ be the numerator of $\text{DDE}_\epsilon$. $\partial_x E_G(u) = 0$ has $k$ distinct solutions in $\overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{\star}}]]$.

**Lemma 2**

The ideal $\langle \mathcal{S}_{\text{dup}} \rangle : \det(\text{Jac}_{\mathcal{S}_{\text{dup}}})^\infty$ is 0-dimensional and contains the minimal poly. of $G$.

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- The equation $\partial_x E_G = 0$ has the form:

$$u^k = \epsilon^k t + t^2 \sum_{i=0}^{k} u^{k-i} \partial_{y_i} Q(F(t,u), \ldots, \Delta^k F(t,u), t^2, u)$$

- Newton's algorithm implies that we find $k$ solutions of the form

$$U_\ell(t) = \epsilon \cdot t^{\frac{1}{k}} \cdot \zeta^\ell + O(t^{\frac{2}{k}}),$$

for $\zeta$ a primitive $k$-th root of unity.

# Sketch of proof (Lemma 2)

**Lemma 2**

The ideal $\langle \mathcal{S}_{\mathrm{dup}} \rangle : \det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}})^{\infty}$ is 0-dimensional and contains the minimal poly. of $G$.

- Recall: $E_G(u) := E_G(F(t,u), F(t,0), \ldots, \partial_u^{k-1} F(t,0), t, u)$ where $E_G$ describes

$$G(t,u,\epsilon) = f(u) + t^2 \cdot Q(G(t,u,\epsilon), \Delta_a G(t,u,\epsilon), \ldots, \Delta_a^k G(t,u,\epsilon), t^2, u) + t\epsilon^k \Delta^k G(t,u,\epsilon).$$

$$\mathcal{S}_{\mathrm{dup}} := \left\{ E_G(u_i) = 0, \partial_x E_G(u_i) = 0, \partial_u E_G(u_i) = 0 : i = 1, \ldots, k \right\}.$$

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$$\mathcal{S}_{\mathrm{dup}} \coloneqq \left\{ E_G(u_i) = 0, \partial_x E_G(u_i) = 0, \partial_u E_G(u_i) = 0 : i = 1, \dots, k \right\}.$$

- Application of Hilbert's Nullstellensatz: Lemma 2 holds if the Jacobian matrix $\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}} \in \overline{\mathbb{Q}(\epsilon)}[\![t^{\frac{1}{\star}}]\!]^{3k\times 3k}$ of $\mathcal{S}_{\mathrm{dup}}$ is invertible at $(x_1, \dots, x_k, u_1, \dots, u_k, z_0, \dots, z_{k-1}) =$

$$(G(t,U_1), \dots, G(t,U_k), U_1(t), \dots, U_k(t), G(t,0), \dots, \partial_u^{k-1} G(t,0)) \in \overline{\mathbb{Q}(\epsilon)}[\![t^{\frac{1}{\star}}]\!]^{3k}.$$

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$$G(t,u,\epsilon) = f(u) + t^2 \cdot Q(G(t,u,\epsilon), \Delta_a G(t,u,\epsilon), \dots, \Delta_a^k G(t,u,\epsilon), t^2, u) + t\epsilon^k \Delta^k G(t,u,\epsilon).$$

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$$(G(t,U_1), \dots, G(t,U_k), U_1(t), \dots, U_k(t), G(t,0), \dots, \partial_u^{k-1}G(t,0)) \in \overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{\star}}]]^{3k}.$$

- Bousquet-Mélou and Jehanne compute the determinant explicitly:

$$\det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}}) = r \cdot \prod_{i<j} (\zeta^i - \zeta^j) \cdot \prod_{j=0}^{k} \left( \partial_x^2 E(U_j) \partial_u^2 E(U_j) - \partial_x \partial_u E(U_j)^2 \right) + O(t^k) \neq 0.$$

# Summary of the proof and algorithm in the scalar case

- Strategy of the proof:
  1. Given a **DDE** for $F(t, u)$ consider the perturbed $\textbf{DDE}_\epsilon$:

$$G(t,u,\epsilon) = f(u) + t^2 \cdot Q(G(t,u,\epsilon), \Delta_a G(t,u,\epsilon), \ldots, \Delta_a^k G(t,u,\epsilon), t^2, u) + t\epsilon^k \Delta^k G(t,u,\epsilon).$$

     and let $E_G$ be the defining polynomial in the the variables $x, z_0, \ldots, z_{k-1}, t, u, \epsilon$.
  2. Prove that $\partial_x E_G(u) = 0$ has $k$ distinct solutions $u = U(t)$ in $\overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{\star}}]]$.
  3. Define $\mathcal{S} = (E, \partial_x E_G, \partial_u E_G)$ and let $\mathcal{S}_{\mathrm{dup}}$ be the duplicated system.
  4. Show that $\langle \mathcal{S}_{\mathrm{dup}} \rangle : \det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}})^\infty$ is 0-dimensional by proving: $\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}}$ is invertible.

- Algorithm:
  1. Define $E_G \in \mathbb{Q}[x, z_0, \ldots, z_{k-1}, t, u, \epsilon]$ as the numerator of of $\textbf{DDE}_\epsilon$.
  2. Compute $\partial_x E_G$ and $\partial_u E_G$. Define $\mathcal{S}_{\mathrm{dup}}$ in $x_1, \ldots, x_n, u_1, \ldots, u_n, z_0, \ldots, z_{k-1}$.
  3. Saturate $\mathcal{S}_{\mathrm{dup}}$ by adding the equation $m \cdot \det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}}) - 1 = 0$ for a variable $m$.
  4. Compute a non-zero element of $\mathcal{S}_{\mathrm{sat}} \cap \mathbb{Q}[z_0, t]$.

# New result: extension for systems

**Theorem (Notarantonio, Y., 2022)**

*Let $n, k \geq 1$ be integers and $f_1, \dots, f_n \in \mathbb{Q}[u]$, $Q_1, \dots, Q_n \in \mathbb{Q}[y_1, \dots, y_{n(k+1)}, t, u]$ be polynomials. Set $\nabla^k F := F, \Delta_a F, \dots, \Delta_a^k F$. Then the system of equations*

$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \dots, \nabla^k F_n, t, u), \\ \quad \vdots \qquad\qquad\qquad \vdots \\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \dots, \nabla^k F_n, t, u) \end{cases}$$

*admits a unique vector of solutions $(F_1, \dots, F_n) \in \mathbb{Q}[u][[t]]^n$, and all its components are algebraic functions over $\mathbb{Q}(t, u)$.*

# Example

**Theorem (Notarantonio, Y., 2022)**

*A system of DDEs with one catalytic variable admits a unique vector of solutions $(F_1, \dots, F_n) \in \mathbb{Q}[u][[t]]^n$, and all its components are algebraic functions over $\mathbb{Q}(t, u)$.*

Example (introduced and solved in [Bonichon,Bousquet-Mélou,Dorbec,Pennarun, 2006]):

$$\begin{cases} F_1(t,u) = 1 + t \cdot \left(u + 2uF_1(t,u)^2 + 2uF_2(t,1) + u\frac{F_1(t,u) - uF_1(t,1)}{u-1}\right), \\ F_2(t,u) = t \cdot \left(2uF_1(t,u)F_2(t,u) + uF_1(t,u) + uF_2(t,1) + u\frac{F_2(t,u) - uF_2(t,1)}{u-1}\right). \end{cases}$$

It holds: $G = F_1(t,1)$ and $F_2(t,1)$ are algebraic functions. For example:

$$64t^3G^3 + 2t\left(24t^2 - 36t + 1\right)G^2 - \left(15t^3 - 9t^2 - 19t + 1\right)G + t^3 + 27t^2 - 19t + 1 = 0.$$

# Generic system case

- The system
$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \dots, \nabla^k F_n, t, u), & \mid \cdot u^{m_1} \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots & \\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \dots, \nabla^k F_n, t, u) & \mid \cdot u^{m_n} \end{cases}$$
defines $n$ polynomial equations $E_1 = 0, \dots, E_n = 0$ in $\mathbb{Q}[x_1, \dots, x_n, z_0, \dots, z_{nk-1}, u, t]$.

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defines $n$ polynomial equations $E_1 = 0, \dots, E_n = 0$ in $\mathbb{Q}[x_1, \dots, x_n, z_0, \dots, z_{nk-1}, u, t]$.
- The "derivative" of $(E_1, \dots, E_n)$ with respect to $u$:

$$\begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_n} E_n \end{pmatrix} \cdot \begin{pmatrix} \partial_u F_1 \\ \vdots \\ \partial_u F_n \end{pmatrix} + \begin{pmatrix} \partial_u E_1 \\ \vdots \\ \partial_u E_n \end{pmatrix} = 0. \qquad (1)$$

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$$\begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_n} E_n \end{pmatrix} \cdot \begin{pmatrix} \partial_u F_1 \\ \vdots \\ \partial_u F_n \end{pmatrix} + \begin{pmatrix} \partial_u E_1 \\ \vdots \\ \partial_u E_n \end{pmatrix} = 0. \qquad (1)$$
- Now define:
$$\mathrm{Det} := \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_n} E_n \end{pmatrix} \quad \text{and} \quad \mathrm{P} := \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_1} E_{n-1} & \partial_{x_1} E_n \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_{n-1}} E_1 & \dots & \partial_{x_{n-1}} E_{n-1} & \partial_{x_{n-1}} E_n \\ \partial_u E_1 & \dots & \partial_u E_{n-1} & \partial_u E_n \end{pmatrix}.$$
- Observation: (1) and $\mathrm{Det} = 0$ imply $\mathrm{P} = 0$.
Therefore we have: $n + 2$ **equations**, and $n + nk + 1$ **variables**.

# Generic system case

- Equations: $\mathcal{S} = \{E_1(u), \ldots, E_n(u), \mathrm{Det}(u), P(u)\}$.
- If $\mathrm{Det}(u) = 0$ has $nk$ distinct solutions in $\overline{\mathbb{Q}}[[t^{\frac{1}{\star}}]]$, then define the system:

$$\mathcal{S}_{\mathrm{dup}} := \begin{cases} E_1(u_i) = E_2(u_i) = \cdots = E_n(u_i) = 0, \\ \mathrm{Det}(u_i) = 0, \\ \mathrm{P}(u_i) = 0. \end{cases} \qquad \text{for } i = 1, \ldots, nk.$$

- In total $(n+2) \cdot nk = nk(n+2)$ equations.
- Variables: $\underbrace{x_1, \ldots, x_{n^2k}}_{F_i(U_j)}, \underbrace{z_0, \ldots, z_{nk-1}}_{\partial^j F_i(t,a)}, \underbrace{u_1, \ldots, u_{nk}}_{U_i} \Rightarrow n^2k + nk + nk = nk(n+2)$.
- $\Rightarrow$ Can hope for a 0-dimensional system.

# Outline of the proof for systems in the general case

1. Given a system of **DDE**s for $F_i(t,u)$ consider the perturbed system $\textbf{DDE}_\epsilon$.

2. Prove that $\mathrm{Det}(u) = 0$ has $nk$ distinct solutions $u = U(t)$ in $\overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{\star}}]]$.

3. Define $\mathcal{S} = (E_1, \dots, E_n, \mathrm{Det}, \mathrm{P})$ and let $\mathcal{S}_{\mathrm{dup}}$ be the duplicated system in $nk(n+2)$ variables.

4. Show: $\langle \mathcal{S}_{\mathrm{dup}} \rangle : \det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}})^\infty$ is 0-dimensional by proving: $\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}}$ is invertible.

# Sketch of proof: Step 1, deformation

$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \dots, \nabla^k F_n, t, u), \\ \quad \vdots \qquad\qquad\qquad \vdots \\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \dots, \nabla^k F_n, t, u) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} G_1 = f_1(u) + t^\alpha \cdot Q_1(\nabla^k G_1, \nabla^k G_2, \dots, \nabla^k G_n, t^\alpha, u) + t \cdot \epsilon^k \cdot \sum_{i=1}^n \gamma_{1,i} \cdot \Delta^k G_i, \\ \quad \vdots \qquad\qquad\qquad \vdots \\ G_n = f_n(u) + t^\alpha \cdot Q_n(\nabla^k G_1, \nabla^k G_2, \dots, \nabla^k G_n, t^\alpha, u) + t \cdot \epsilon^k \cdot \sum_{i=1}^n \gamma_{n,i} \cdot \Delta^k G_i, \end{cases}$$

where $\gamma_{i,i} = i^k$ and $\gamma_{i,j} = t^\beta$, and $\alpha \gg \beta \gg 0$ are chosen sufficiently large.

- It holds: $G_i(t, u, \epsilon)$ algebraic over $\mathbb{Q}(t, u, \epsilon) \Rightarrow F_i(t, u)$ algebraic over $\mathbb{Q}(t, u)$.
- After multiplication by $u_i^{m_i}$, each equation for $G_i$ induces a polynomial equation:

$$E_i(\nabla^k G_1, \nabla^k G_2, \dots, \nabla^k G_n, t, u, \epsilon) = 0, \quad i = 1, \dots, n.$$

# Step 2: Definition of Det($u$) and proof of $nk$ distinct solutions

$$\mathrm{Det} := \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_n} E_n \end{pmatrix}$$

**Lemma 1**

The equation $\mathrm{Det}(u) = 0$ has $nk$ distinct solutions in $\overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{\star}}]]$

# Step 2: Definition of Det($u$) and proof of $nk$ distinct solutions

$$\mathrm{Det} := \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_n} E_n \end{pmatrix}$$

**Lemma 1**

The equation $\mathrm{Det}(u) = 0$ has $nk$ distinct solutions in $\overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{\star}}]]$

- We have

$$\mathrm{Det}(u) = \det \begin{pmatrix} -u^{m_1} + t\epsilon^k \gamma_{1,1} u^{m_1-k} & \cdots & t\epsilon^k \gamma_{1,n} u^{m_1-k} \\ \vdots & \ddots & \vdots \\ t\epsilon^k \gamma_{n,1} u^{m_n-k} & \cdots & -u^{m_n} + t\epsilon^k \gamma_{n,n} u^{m_n-k} \end{pmatrix} + O(t^\alpha u^{m_1+\cdots+m_n-nk}).$$

- Using $\gamma_{i,i} = i^k$ and $\gamma_{i,j} = t^\beta$, we get $\mathrm{Det} = \prod_{j=1}^n (-u^k + t\epsilon^k j^k) \mod t^{n+1}$.
- Newton's algorithm:

$$U_i = \zeta^\ell \cdot j \cdot t^{\frac{1}{k}} + O(t^{\frac{2}{k}}), \quad \text{for } \ell = 1, \dots, k \text{ and } j = 1, \dots, n, \text{ with } \zeta^k = 1.$$

# Outline of the proof for systems in the general case

1. Given a system of **DDE**s for $F_i(t,u)$ consider the perturbed system $\mathbf{DDE}_\epsilon$.
2. Prove that $\mathrm{Det}(u) = 0$ has $nk$ distinct solutions $u = U(t)$ in $\overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{\star}}]]$.
3. Define $\mathcal{S} = (E_1, \ldots, E_n, \mathrm{Det}, \mathrm{P})$ and let $\mathcal{S}_{\mathrm{dup}}$ be the duplicated system in $nk(n+2)$ variables.
4. Show: $\langle \mathcal{S}_{\mathrm{dup}} \rangle : \det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}})^\infty$ is 0-dimensional by proving: $\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}}$ is invertible.

# Step 4: $\mathrm{Jac}_{\mathcal{S}_{\text{dup}}}$ is invertible (very brief sketch)

$$\mathrm{Jac}_{\mathcal{S}_{\text{dup}}} = \begin{pmatrix} A_1 & & 0 & B_1 \\ & \ddots & & \vdots \\ 0 & & A_{nk} & B_{nk} \end{pmatrix} \in \overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{\star}}]]^{nk(n+2)\times nk(n+2)},$$

$$A_i := \begin{pmatrix} \partial_{x_1} E_1^{(i)}(U_i) & \dots & \partial_{x_n} E_1^{(i)}(U_i) & \partial_{u_i} E_1^{(i)}(U_i) \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} E_n^{(i)}(U_i) & \dots & \partial_{x_n} E_n^{(i)}(U_i) & \partial_{u_i} E_n^{(i)}(U_i) \\ \partial_{x_1} \mathrm{Det}^{(i)}(U_i) & \dots & \partial_{x_n} \mathrm{Det}^{(i)}(U_i) & \partial_{u_i} \mathrm{Det}^{(i)}(U_i) \\ \partial_{x_1} P^{(i)}(U_i) & \dots & \partial_{x_n} P^{(i)}(U_i) & \partial_{u_i} P^{(i)}(U_i) \end{pmatrix}, B_i := \begin{pmatrix} \partial_{z_0} E_1^{(i)}(U_i) & \dots & \partial_{z_{nk-1}} E_1^{(i)}(U_i) \\ \vdots & \ddots & \vdots \\ \partial_{z_0} E_n^{(i)}(U_i) & \dots & \partial_{z_{nk-1}} E_n^{(i)}(U_i) \\ \partial_{z_0} \mathrm{Det}^{(i)}(U_i) & \dots & \partial_{z_{nk-1}} \mathrm{Det}^{(i)}(U_i) \\ \partial_{z_0} P^{(i)}(U_i) & \dots & \partial_{z_{nk-1}} P^{(i)}(U_i) \end{pmatrix}.$$

$$\Rightarrow \det(\mathrm{Jac}_{\mathcal{S}_{\text{dup}}}) = \pm\Big(\prod_{i=1}^{nk} \det(\mathrm{Jac}_i(U_i))\Big) \cdot \det(\Lambda), \quad \text{for}$$

$$\mathrm{Jac}_i(u) \in \mathbb{Q}(\epsilon)[u][[t]]^{(n+1)\times(n+1)} \text{ and } \Lambda \in \overline{\mathbb{Q}(\epsilon)}[[t^{\frac{1}{\star}}]]^{nk\times nk}$$

**Method:** Analyze the (lowest) valuation in $t$ to show non-vanishing.

# Special Eulerian planar orientations

- [Bonichon, Bousquet-Mélou, Dorbec, Pennarun, 2006] consider and solve:

$$\begin{cases} F_1(t,u) = 1 + t \cdot \left(u + 2uF_1(t,u)^2 + 2uF_2(t,1) + u\frac{F_1(t,u)-uF_1(t,1)}{u-1}\right), \\ F_2(t,u) = t \cdot \left(2uF_1(t,u)F_2(t,u) + uF_1(t,u) + uF_2(t,1) + u\frac{F_2(t,u)-uF_2(t,1)}{u-1}\right). \end{cases}$$

# Special Eulerian planar orientations

- We get polynomial equations

$$\begin{cases} E_1 = (1-x_1)\cdot(u-1) + t\cdot(2u^2x_1^2 - u^2z_0 + 2u^2z_1 - 2ux_1^2 + u^2 + ux_1 - 2uz_1 - u), \\ E_2 = x_2\cdot(1-u) + t\cdot(2u^2x_1x_2 + u^2x_1 - 2ux_1x_2 - ux_1 + ux_2 - uz_1). \end{cases}$$

- Then define

$$\begin{cases} \mathrm{Det} = (4tu^2x_1 - 4tux_1 + tu - u + 1)(2tu^2x_1 - 2tux_1 + tu - u + 1), \\ \mathrm{P} = -2tx_1x_2 - tx_1 + tx_2 - tz_1 - x_2 + P_1\cdot u + P_2\cdot u^2 + P_3\cdot u^3, \end{cases}$$

- $\mathcal{S}_{\mathrm{dup}} =$
$\big(E_1(x_1, x_2, z_0, z_1, u_1), E_2(x_1, x_2, z_0, z_1, u_1), \mathrm{Det}(x_1, x_2, z_0, z_1, u_1), \mathrm{P}(x_1, x_2, z_0, z_1, u_1),$
$(E_1(x_3, x_4, z_0, z_1, u_2), E_2(x_3, x_4, z_0, z_1, u_2), \mathrm{Det}(x_3, x_4, z_0, z_1, u_2), \mathrm{P}(x_3, x_4, z_0, z_1, u_2)\big)$
- Compute a generator of $\langle \mathcal{S}_{\mathrm{dup}}, m\cdot(u_1-u_2)-1\rangle \cap \mathbb{Q}[z_0, t]$.

# Special Eulerian planar orientations in Maple

```
E1 := numer(-x1+1 + 2*t*u*x1^2 + 2*t*u*z1 + t*u*(-u*z0+x1)/(u-1)+t*u);
E2 := numer(-x2 + 2*t*u*x1*x2 + t*u*x1 + t*u*z1 + t*u*(-u*z1+x2)/(u-1));
Jac := Matrix([[diff(E1, x1),diff(E1, x2)],[diff(E2, x1),diff(E2, x2)]]);
Det := LinearAlgebra[Determinant](Jac);
Pm := Matrix([[diff(E1, x1), diff(E2, x1)], [diff(E1, u), diff(E2, u)]]);
P := LinearAlgebra[Determinant](Pm);
S := [E1, E2, det, P];
S1 := op(subs(x1=x1,x2=x2,u=u1,S);
S2 := op(subs(x1=x3,x2=x4,u=u2,S);
Sdup := [S1,S2, m*(u1 - u2) - 1];
G := polynomial_elimination(Sdup, z0, t);
```

$$(z0-1)(2tz0+t-1)\left(64t^3z0^3+48t^3z0^2-15t^3z0-72t^2z0^2+t^3+9t^2z0+\cdots\right)\cdots$$

# Summary and conclusion

- **Systems of DDEs** with one catalytic variable have an algebraic solution.
- There exists an algorithm for finding minimal polynomials of such solutions.
- Currently ongoing work on improving the efficiency and effective handling of more catalytic variables.