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Declaration of Academic Honesty

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Abstract

This thesis is about a very famous problem in number theory: The estimation of the amount of prime numbers lower than a given value: The so-called Prime Number Theorem (PNT). It turns out that to answer this question one can (or even should) use deep results of complex analysis and so this work is more about this topic in mathematics. This thesis proves the PNT assuming the readers fundamental knowledge of complex analysis – theorems as “Cauchy’s Theorem”, “Morera’s Theorem”, etc. since it is needed to understand many of the theorems and proofs given in this thesis.

Introduction

Some call the “prime numbers” the building blocks of natural numbers as they play a huge role in understanding number theory and all questions connected with it. This fundamental realization was already made by Euclid in his famous book "Elements". He proved there that any natural number can be uniquely written as a product of primes and that there are infinitely many of them.
As time went by more and more interesting and surprising results were discovered about primes and natural numbers, but only one statement became known as "The Prime Number Theorem". It states that the amount of prime numbers smaller or equal to x is (asymptotically) equal to $\frac{x}{\ln x}$. L.Euler, C.F.Gauss, B.Riemann and many, many other very famous mathematicians contributed to the proof of this elegant statement. Exactly this theorem and its proof are the topics of this thesis.

Number theory is a special branch in mathematics. It is not only one of the oldest, it also became known as the toughest: No other area in this science has so many unsolved but easy-looking problems. For example, it is still open if there are finite of infinitely many prime numbers which differ by exactly two.
Over thousands of years mathematicians figured out many ways to tackle problems in number theory. One concept goes like this: transfer the problem into another topic in mathematics and solve it there with other, typical for it, methods. So, as we will see that the proof of the Prime Number Theorem presented in this work is actually more a complex analysis problem which relies on studying the behavior of a complex function, called the zeta function.
In the next chapter the reader will find historic information about this function and then a whole section devoted to the explanation that this function is well-defined on $\mathbb{C}$ and to its properties. In the last chapter of this thesis I give historic notes about the Prime Number Theorem and finally provide the proof for it.

Contents

Chapter 1

The Riemann zeta function

In this chapter we introduce the so-called zeta function and discuss fundamental properties of it. But to do so we must begin with the less interesting gamma function because as we will see these two functions are closely connected to each other.

The reader may wonder how this analytic function, its poles and growth are necessary for proving a theorem about prime numbers. This is a legitimate question that will hopefully be answered in this or next the chapter.

1.1 Properties of the gamma- and zeta functions

1.1.1 The gamma function

We begin with a simple definition of notation:

Definition. $\mathbb{H}_\alpha := \{z \in \mathbb{C} : \mathrm{Re}(z) > \alpha\}$

On an open set like this we can now define a function:

Definition. For $z \in \mathbb{H}_0$ we define the gamma function as:

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$$

To prove that this function is well-defined we have to show that this integral converges on $\mathbb{H}_0$. For $z = \sigma + iy$ such that $\sigma > 0$ we choose T so large that the inequality $(\sigma - 1) \ln t \leq t/2$ holds for any $t \geq T$. Then, because $e^{-t} \leq 1$ for any $t \geq 0$:

$$\begin{aligned}|\Gamma(z)| &= \left|\int_0^\infty t^{z-1}e^{-t}\mathrm{d}t\right| \leq \int_0^\infty |t^{z-1}e^{-t}|\mathrm{d}t = \int_0^\infty |t^{z-1}|e^{-t}\mathrm{d}t = \int_0^\infty t^{\sigma-1}e^{-t}\mathrm{d}t \\ &= \int_0^T t^{\sigma-1}e^{-t}\mathrm{d}t + \int_T^\infty t^{\sigma-1}e^{-t}\mathrm{d}t \leq \int_0^T t^{\sigma-1}\mathrm{d}t + \int_T^\infty e^{(\sigma-1)\ln t - t}\mathrm{d}t \\ &\leq \frac{T^\sigma}{\sigma} + \int_T^\infty e^{t/2-t}\mathrm{d}t = \frac{T^\sigma}{\sigma} + 2e^{-T/2} < \infty.\end{aligned}$$

So for every $z \in \mathbb{H}_0$ our function is well-defined.
Now we will discuss and prove some essential properties of this function.

Lemma 1.1.1. $\Gamma(z)$ *is holomorphic on* $\mathbb{H}_0$*.*

Proof. First we define $f_n(z) := \int_{1/n}^n t^{z-1}e^{-t}\mathrm{d}t$ for $z \in \mathbb{H}_0$. Our goal is now to show that each f_n is holomorphic on $\mathbb{H}_0$ and that $f_n \to \Gamma$ uniformly on compact subsets of $\mathbb{H}_0$. This would already imply that Γ is holomorphic.
To prove that f_n is holomorphic we will use Morera's Theorem. Let $\gamma : [0,1] \to \mathbb{C}$ be a closed C^1 curve on $\mathbb{H}_0$. We can compute the following integral using Fubini's Theorem:

$$\begin{aligned}\oint_\gamma f_n(z)\mathrm{d}z &= \oint_\gamma \int_{1/n}^n t^{z-1}e^{-t}\mathrm{d}t\mathrm{d}z = \int_0^1 \int_{1/n}^n t^{\gamma(s)-1}e^{-t}\gamma'(s)\mathrm{d}t\mathrm{d}s \\ &= \int_{1/n}^n \int_0^1 t^{\gamma(s)-1}e^{-t}\gamma'(s)\mathrm{d}s\mathrm{d}t = \int_{1/n}^n \oint_\gamma t^{z-1}e^{-t}\mathrm{d}z\mathrm{d}t = 0,\end{aligned}$$

since $z \mapsto t^{z-1}e^{-t}$ is holomorphic on $\mathbb{H}_0$ for positive t. Thus each f_n is holomorphic. To justify the uniform convergence let K be a compact subset of $\mathbb{H}_0$. Also let $\sigma_1 := \min\{\mathrm{Re}(z) : z \in K\}$ and $\sigma_2 := \max\{\mathrm{Re}(z) : z \in K\}$. Now we can assess for $z \in K$:

$$\begin{aligned}|f_n(z) - \Gamma(z)| &= \left|\int_{1/n}^n t^{z-1}e^{-t}\mathrm{d}t - \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t\right| \\ &= \left|\int_0^{1/n} t^{z-1}e^{-t}\mathrm{d}t + \int_n^\infty t^{z-1}e^{-t}\mathrm{d}t\right| \\ &\leq \int_0^{1/n} |t^{z-1}e^{-t}|\mathrm{d}t + \int_n^\infty |t^{z-1}e^{-t}|\mathrm{d}t \\ &\leq \int_0^{1/n} t^{\sigma_1-1}e^{-t}\mathrm{d}t + \int_n^\infty t^{\sigma_2-1}e^{-t}\mathrm{d}t.\end{aligned}$$

Both integrals tend to 0 as $n \to \infty$ independently of $z \in K$. This finishes the proof. ∎

Now we want to extend the gamma function meromorphically to $\mathbb{C}$. To do so we start with proving the following lemma:

Lemma 1.1.2. *For $z \in \mathbb{H}_0$ it holds:*

$$\Gamma(z+1) = z\Gamma(z) \tag{1.1.1}$$

And as consequences for $z \in \mathbb{H}_0$ and $n \in \mathbb{N}_0$:

$$\Gamma(z+n+1) = (z+n)(z+n-1)\cdots z\Gamma(z) \tag{1.1.2}$$

$$\Gamma(n+1) = n! \tag{1.1.3}$$

Proof. Equation (1.1.1) can be easily shown by integration by parts:

$$\begin{aligned}\Gamma(z+1) = \int_0^\infty t^{(z+1)-1}e^{-t}\mathrm{d}t &= \int_0^\infty t^z e^{-t}\mathrm{d}t \\ &= -t^z e^{-t}\Big|_0^\infty + \int_0^\infty z t^{z-1}e^{-t}\mathrm{d}t = 0 + z\Gamma(z)\end{aligned}$$

To show *Equation* (1.1.2) we just iterate *Equation* (1.1.1) exactly n times:

$$\begin{aligned}\Gamma(z+n+1) &= (z+n)\Gamma(z+n) = (z+n)(z+n-1)\Gamma(z+n-1) \\ &= ... = (z+n)(z+n-1)\cdots z\Gamma(z)\end{aligned}$$

Finally by putting $z = 1$ in *Equation* (1.1.2) and using the fact that $\Gamma(1) = \int_0^\infty t^0 e^{-t}\mathrm{d}t = 1$ we get for $n \in \mathbb{N}_0$:

$$\Gamma(n+2) = (n+1)!$$

Again with the fact that $\Gamma(1) = 1 = 0!$ we see that the last claim is also true. $\square$

Now we come to the theorem that guarantees the unique meromorphical extension:

Theorem 1.1.3. *There exists a unique continuation of the gamma function as a meromorphic function on $\mathbb{C}$, where Γ has only simple poles on negative natural numbers and zero. Furthermore $\mathrm{Res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}$, $n \in \mathbb{N}$.*

Proof. We prove that for every $m \in \mathbb{N}_0$ there exists a unique meromorphic continuation of Γ on $\mathbb{H}_{-m}$ which is holomorphic on $\mathbb{H}_{-m}\setminus\{0,-1,-2,...,-m+1\}$ by induction on m:
For $m = 0$ we obviously just have the gamma function on $\mathbb{H}_0$.
Assume now that the claim is true for every $m < M \in \mathbb{N}$. We can define

$$\Gamma_M(z) := \frac{\Gamma(z+M)}{z(z+1)\cdots(z-M+1)}.$$

First we see that the right-hand side is well defined and meromorphic on $\mathbb{H}_{-M}$

with simple poles on $z = 0, 1, ..., M-1$, since $z + M \in \mathbb{H}_0$ if $z \in \mathbb{H}_{-M}$; so must the left-hand side. As $\Gamma(z)$ (or its continuation on $\mathbb{H}_m$) is holomorphic on any $\mathbb{H}_{-m} \setminus \{-1, -2, ..., -m+1\}$ we have that $\Gamma_M(z)$ is holomorphic on $\mathbb{H}_{-M} \setminus \{-1, -2, ..., -M+1\}$. Now if $z \in H_0$ we have because of *Equation* (1.1.2) $\Gamma_M(z) \equiv \Gamma(z)$. And so because of the Identity Theorem ($\mathbb{H}_{-M} \cap \mathbb{H}_0 = \mathbb{H}_0$) $\Gamma_M(z)$ is the unique continuation of $\Gamma(z)$ on $\mathbb{H}_{-M}$. This completes the induction.
For now we call this unique continuation $\widetilde{\Gamma}$ and compute its residues for $-n$ where $n \in \mathbb{N}_0$

$$\begin{aligned}\mathrm{Res}_{z=-n}\widetilde{\Gamma}(z) &= \lim_{z \to -n} (z+n)\widetilde{\Gamma}(z) = \lim_{z \to -n} \frac{\cancel{(z+n)}\Gamma(z+n+1)}{z(z+1)\cdots\cancel{(z+n)}} \\ &= \frac{\Gamma(1)}{(-n)(-n+1)\cdots(-n+n-1)} = (-1)^n \frac{1}{n!}\end{aligned}$$

□

Remark. We proved the existence and uniqueness of a continuation of Γ on $\mathbb{C}$, so from now on we will refer to this continuation whenever we use "$\Gamma(z)$".

The last but very important property of our function is the connection to the sine function. To prove the next theorem we will need this lemma:

Lemma 1.1.4. *For $0 < a < 1$ the following identity is true:*

$$\int_0^\infty \frac{x^{a-1}}{1+x} \mathrm{d}x = \frac{\pi}{\sin \pi a} \tag{1.1.4}$$

Proof. The integral we have to compute has the form $\int_0^\infty x^\lambda R(x)dx$ where $\lambda = a$ and $R(x) = \frac{1}{x(x+1)}$. To compute it we use the formula proven in [4] on page 83 at "Anwendung 4":
Let $0 < \lambda < 1$ and $R(x)$ be a rational function that has with a double zero at infinity and has a simple pole at 0. Then:

$$\int_0^\infty x^\lambda R(x) \mathrm{d}x = \frac{2\pi i}{1 - e^{2\pi i \lambda}} \sum_{a \neq 0} \mathrm{Res}_{z=a} z^\lambda R(z). \tag{1.1.5}$$

Note that the logarithm here is defined on the positive sliced complex plane $G := \mathbb{C} \setminus \mathbb{R}_0^+$ by $\ln(re^{i\varphi}) := \ln r + i\varphi$ for $r > 0$ and $0 < \varphi < 2\pi$. Then for this logarithm $\ln : G \to \mathbb{C}$ we define $z^\lambda := e^{\lambda \ln z}$.
Now, as all conditions perfectly apply in our case and the only residue that

matters is at $z = -1$ we simply compute:

$$\begin{aligned}\int_0^\infty \frac{x^{a-1}}{1+x}\mathrm{d}x &= \int_0^\infty x^a \frac{1}{x(1+x)}\mathrm{d}x = \frac{2\pi i}{1-e^{2\pi i a}}\mathrm{Res}_{z=-1} z^a \frac{1}{z(1+z)} \\ &= \frac{2\pi i}{1-e^{2\pi i a}} \frac{e^{\ln(z)a}}{1+2z}\bigg|_{z=-1} = \frac{2\pi i}{1-e^{2\pi i a}} \frac{e^{i\pi a}}{-1} \\ &= \frac{2\pi i}{e^{i\pi a} - e^{-i\pi a}} = \frac{\pi}{\sin \pi a}\end{aligned}$$

□

Using this lemma we are now able to prove the theorem:

Theorem 1.1.5. *For all* $z \in \mathbb{C}$,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{1.1.6}$$

Proof. Observe that $\Gamma(1-z)$ is meromorphic with simple poles at positive integers, so the function $z \mapsto \Gamma(z)\Gamma(1-z)$ has a pole if and only if $z \in \mathbb{Z}$, a property also shared by $\frac{\pi}{\sin \pi z}$.
Because of the Identity Theorem it suffices to prove *Equation* (1.1.6) for $z \in \mathbb{R}$ and even $0 < z < 1$, which we will assume from now on.
Now note that for $v > 0$ we can do the transformation $t = uv$ to get:

$$\Gamma(1-z) = \int_0^\infty t^{-z}e^{-t}\mathrm{d}t = v\int_0^\infty (uv)^{-z}e^{-uv}\mathrm{d}u.$$

This and Fubini's Theorem give us:

$$\begin{aligned}\Gamma(z)\Gamma(1-z) &= \int_0^\infty v^{z-1}e^{-v}\Gamma(1-z)\mathrm{d}v \\ &= \int_0^\infty v^{z-1}e^{-v}\left(v\int_0^\infty (uv)^{-z}e^{-uv}\mathrm{d}u\right)\mathrm{d}v \\ &= \int_0^\infty \int_0^\infty e^{-v(1+u)}u^{-z}\mathrm{d}u\mathrm{d}v \\ &= \int_0^\infty \frac{u^{-z}}{1+u}\mathrm{d}u = \frac{\pi}{\sin \pi(1-z)} = \frac{\pi}{\sin \pi z}\end{aligned}$$

□

Rewriting this theorem gives us $\frac{1}{\Gamma(z)} = \Gamma(1-z)\frac{\sin \pi z}{\pi}$. We know that $\sin(\pi z)$ has only simple zeros at whole numbers and $\Gamma(1-z)$ has simple poles at positive integers. Multiplication of these functions cancels the poles to removable ones and we get:

Theorem 1.1.6. *The function $z \mapsto \frac{1}{\Gamma(z)}$ is entire and has only simple zeros at $z = 0, -1, -2, \ldots$.*

The gamma function has a lot more interesting properties many of which the interested reader can find in [5], but now have enough information about it to discuss the fundamental properties of the zeta function. We will see soon how closely connected these functions are.

1.1.2 The zeta function

Similarly to the gamma function we will start defining the zeta function only on a halfplane - in this case $\mathbb{H}_1$ - and then extending it uniquely and meromorphically on $\mathbb{C}$.

Definition. For $s \in \mathbb{H}_1$ we define:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

We have to prove that on $\mathbb{H}_1$ this sum converges. Then the function is well-defined there. We prove an even more powerful result:

Proposition 1.1.7. *The sum defining the zeta function converges on $\mathbb{H}_1$ and ζ is holomorphic on this half-plane.*

Proof. For $s \in \mathbb{H}_1$ and $N \in \mathbb{N}$ we define $\zeta_N(s)$ by the partial sum:

$$\zeta_N(s) := \sum_{n=1}^{N} \frac{1}{n^s}.$$

Of course $\zeta_N(s)$ is holomorphic on $\mathbb{H}_1$ for any $N \in \mathbb{N}$ as it is the sum of finitely many holomorphic functions. We show that $\zeta_N(s) \to \zeta(s)$ uniformly on $\mathbb{H}_1$, then the proposition is proven. So let $\varepsilon > 0$, then for $s \in \mathbb{H}_{1+\varepsilon}$:

$$\begin{aligned}\left|\zeta_N(s) - \zeta(s)\right| &= \left|\sum_{n=1}^{N} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{n^s}\right| = \left|\sum_{n=N+1}^{\infty} \frac{1}{n^s}\right| \leq \sum_{n=N+1}^{\infty} \left|\frac{1}{n^s}\right| \\ &= \sum_{n=N+1}^{\infty} \frac{1}{n^\sigma} < \sum_{n=N+1}^{\infty} \frac{1}{n^{1+\varepsilon}} < \int_{N+1}^{\infty} \frac{1}{(t-1)^{1+\varepsilon}} \mathrm{d}t = \frac{1}{\varepsilon N^\varepsilon} \overset{N\to\infty}{\longrightarrow} 0,\end{aligned}$$

independently from $s \in \mathbb{H}_{1+\varepsilon}$ for any arbitrary small ε, so $\zeta_N(s) \to \zeta(s)$ uniformly on $\mathbb{H}_1$. □

Our goal is to prove that there exists a meromorphic continuation of $\zeta(s)$ on $\mathbb{C}$. This is actually a very deep result in mathematics which is pretty difficult to show. We start with a lemma:

Lemma 1.1.8. *For $s \in \mathbb{H}_1$ it holds:*

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1}\mathrm{d}t. \tag{1.1.7}$$

Proof. We know that for $s \in \mathbb{H}_0$ is true:

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\mathrm{d}t,$$

and we make the variable transformation: $t = nu$ for a $n \in \mathbb{N}$. This gives us:

$$\Gamma(s) = \int_0^\infty n(nu)^{s-1}e^{-nu}\mathrm{d}u,$$

which is equivalent to:

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty u^{s-1}e^{-nu}\mathrm{d}u.$$

So we can now compute for $s \in \mathbb{H}_1$:

$$\begin{aligned}
\zeta(s)\Gamma(s) &= \left(\sum_{n=1}^\infty \frac{1}{n^s}\right) \cdot \Gamma(s) = \sum_{n=1}^\infty \frac{\Gamma(s)}{n^s} = \sum_{n=1}^\infty \int_0^\infty u^{s-1}e^{-nu}\mathrm{d}u \\
&= \int_0^\infty u^{s-1}\left(\sum_{n=1}^\infty (e^{-u})^n\right)\mathrm{d}u = \int_0^\infty u^{s-1}\frac{e^{-u}}{1-e^{-u}}\mathrm{d}u \\
&= \int_0^\infty \frac{u^{s-1}}{e^u - 1}\mathrm{d}u
\end{aligned}$$

□

This equation tells us that to understand the zeta function we have to study study this integral. The main idea of the following proof is taken from [2]: first we split it two parts which we will discuss separately:

$$\int_0^\infty \frac{t^{z-1}}{e^t-1}\mathrm{d}t = \int_0^1 \frac{t^{z-1}}{e^t-1}\mathrm{d}t + \int_1^\infty \frac{t^{z-1}}{e^t-1}\mathrm{d}t$$

We deal with the second integral first. This integral converges on $\mathbb{C}$, because we can assess it using the inequality $e^t - 1 > e^{t-1}$ for $t > 1$ and so for $s = \sigma + it$ we get:

$$\left|\int_1^\infty \frac{t^{z-1}}{e^t-1}\mathrm{d}t\right| \leq \int_1^\infty \frac{|t^{z-1}|}{e^t-1}\mathrm{d}t < \int_1^\infty \frac{t^{\sigma-1}}{e^{t-1}}\mathrm{d}t = e\int_1^\infty t^{\sigma-1}e^{-t}\mathrm{d}t.$$

Now we distinguish between two cases:

If $\sigma < 1$ then $t^{\sigma-1} < 1$ and:

$$e\int_1^\infty t^{\sigma-1}e^{-t}\mathrm{d}t < e\int_1^\infty e^{-t}\mathrm{d}t = 1.$$

If $\sigma \geq 1$ then:

$$e\int_1^\infty t^{\sigma-1}e^{-t}\mathrm{d}t < e\int_0^\infty t^{\sigma-1}e^{-t}\mathrm{d}t = e\Gamma(\sigma) < \infty.$$

We see in both cases that the integral converges. So the following function is well-defined:

$$F(z) := \int_1^\infty \frac{t^{z-1}}{e^t-1}\mathrm{d}t.$$

To prove that $F(z)$ is holomorphic on $\mathbb{C}$ we define (similar to the proof that $\Gamma(z)$ is holomorphic on $\mathbb{H}_0$):

$$F_n(z) := \int_1^n \frac{t^{z-1}}{e^t-1}\mathrm{d}t,$$

and prove that each F_n is holomorphic and that $F_n \to F$ uniformly on any compact set $K \subset \mathbb{C}$.
To show that each F_n is holomorphic we use Morera's Theorem and Fubini. So let $\gamma : [0,1] \to \mathbb{C}$ be a closed C^1 curve. Then:

$$\begin{aligned}\oint_\gamma F_n(z)\mathrm{d}z &= \oint_\gamma \int_1^n \frac{t^{z-1}}{e^t-1}\mathrm{d}t\mathrm{d}z = \int_0^1\int_1^n \frac{t^{\gamma(s)-1}}{e^t-1}\gamma'(s)\mathrm{d}t\mathrm{d}s\\ &= \int_1^n\int_0^1 \frac{t^{\gamma(s)-1}}{e^t-1}\gamma'(s)\mathrm{d}s\mathrm{d}t = \int_1^n \frac{1}{e^t-1}\oint_\gamma t^{z-1}\mathrm{d}z\mathrm{d}t = 0,\end{aligned}$$

since the function $z \mapsto t^{z-1}$ is holomorphic for $t \geq 1$.

To prove that $F_n \to F$ uniformly on any compact set $K \subset \mathbb{C}$ we define for a given K $\sigma := \max\{\mathrm{Re}(z) : z \in K\}$ and then for $z \in K$ we compute:

$$\begin{aligned}\left|F_n(z) - F(z)\right| &= \left|\int_1^n \frac{t^{z-1}}{e^t-1}\mathrm{d}t - \int_1^\infty \frac{t^{z-1}}{e^t-1}\mathrm{d}t\right|\\ &\leq \int_n^\infty \frac{|t^{z-1}|}{e^t-1}\mathrm{d}t \leq \int_n^\infty \frac{t^{\sigma-1}}{e^t-1}\mathrm{d}t \overset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

independently from $z \in K$. So $F(z)$ is indeed holomorphic.

To deal with the first integral we note that the function $z \mapsto \frac{1}{e^z-1}$ has simple poles at $z = 2m\pi i$ for $m \in \mathbb{Z}$ each with residue 1. Therefore we can write:

$$\frac{1}{e^z-1} = \frac{1}{z} + G(z),$$

for a function $G(z)$ that is meromorhic with simple poles at $z = 2m\pi i$ for $m \in \mathbb{Z}$ and $m \neq 0$. So on the circle $|z| < 2\pi$ we see that $G(z)$ is holoporphic and we have:

$$\frac{1}{e^z - 1} = \frac{1}{z} + \sum_{n=0}^{\infty} c_n z^n. \tag{1.1.8}$$

By Cauchy estimates, if we fix $0 < r < 2\pi$, it follows that $|c_n| \leq \frac{M}{r^n}$ for some $M \in \mathbb{R}^+$. In particular there exists $M > 0$ such that $|c_n| \leq \frac{M}{2^n}$ for all $n \in \mathbb{N}_0$. Now we use *Equation* (1.1.8) for $t \in [0, 1]$ and so, as for $s \in \mathbb{H}_1$ the series below converges uniformly, we have on this half-plane:

$$\begin{aligned}\int_0^1 \frac{t^{s-1}}{e^t - 1}\mathrm{d}t &= \int_0^1 \left(\frac{t^{s-1}}{t} + \sum_{n=0}^{\infty} t^{s-1} c_n t^n\right)\mathrm{d}t = \int_0^1 t^{s-2}\mathrm{d}t + \int_0^1 \left(\sum_{n=0}^{\infty} c_n t^{s+n-1}\right)\mathrm{d}t \\ &= \frac{1}{s-1} + \sum_{n=0}^{\infty}\left(c_n \int_0^1 t^{s+n-1}\mathrm{d}t\right) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{c_n}{s+n}\end{aligned}$$

We will call now

$$f(z) := \sum_{n=0}^{\infty} \frac{c_n}{z+n} \tag{1.1.9}$$

and prove that for any $R \in \mathbb{R}^+$ this function is meromorphic on the disc $\{z : |z| < R\}$. So let $R \in \mathbb{R}^+$ be arbitrary. Then for $|z| < R$ we can write $f(z)$ as:

$$f(z) = \sum_{n=0}^{\infty} \frac{c_n}{z+n} = \sum_{\substack{n \leq R \\ n \in \mathbb{N}}} \frac{c_n}{z+n} + \sum_{\substack{n > R \\ n \in \mathbb{N}}} \frac{c_n}{z+n} =: f_R^I(z) + f_R^{II}(z).$$

$f^I(z)$ is clearly meromorphic as it is a finite sum of rational functions. Moreover we easily see that it has a (simple) pole for any $z = -n$, whenever n is natural, smaller than R and $c_n \neq 0$.

$f^{II}(z)$ is holomorpic. To justify this claim we first donate n_0 as the smallest integer bigger than R and then for $N \in \mathbb{N}$ and $N > n_0$ we define $f_N(z) := \sum_{n=n_0}^{N} \frac{c_n}{z+n}$. We observe that $f_N(z)$ is holomorphic for each N, since $|z| < R$ together with the triangle inequality implies that $\left|\frac{1}{z+n}\right| < \frac{1}{n-R}$ and so it is is a sum of finitely many holomorphic functions. Furthermore we claim that $f_N \to f_R^{II}$ uniformly on our disc. Remember that we can assess $|c_n| \leq \frac{M}{2^n}$, which means that $\sum_{n \geq 0} |c_n| \leq 2M$ and so it follows:

$$\begin{aligned}|f_N(z) - f_R^{II}(z)| &= \left|\sum_{n=N+1}^{\infty} \frac{c_n}{z+n}\right| \leq \sum_{n=N+1}^{\infty} \frac{|c_n|}{|z+n|} \leq \sum_{n=N+1}^{\infty} \frac{|c_n|}{n-R} \\ &\leq \frac{1}{N+1-R} \sum_{n=N+1}^{\infty} |c_n| \leq \frac{2M}{N+1-R} \xrightarrow{N\to\infty} 0,\end{aligned}$$

independently from z in our disc.
So we have for any $R \in \mathbb{R}^+$ that $f_R^{II}(z)$ is meromorphic on the disc $\{z : |z| < R\}$ with poles on $z \in \{-n : n \in \mathbb{N}_0, n < R\}$; all of those are either simple or removable depending on whether c_n is equal to zero or not. This means that $f(z)$ must be meromorphic on $\mathbb{C}$ with the same position and type of singularities.

Altogether we now have for $z \in \mathbb{H}_1$:

$$\int_0^\infty \frac{t^{z-1}}{e^t - 1}\mathrm{d}t = F(z) + \frac{1}{z-1} + f(z),$$

where $F(z)$ is entire, $\frac{1}{z-1}$ has the simple pole at $z = 1$ and $f(z)$ is also meromorphic on $\mathbb{C}$ such that the poles of $f(z)$ are either simple or removable, but all lie on $z \in \{-n : n \in \mathbb{N}_0\}$. This means we found an (the) analytic continuation of $z \mapsto \int_0^\infty \frac{t^{z-1}}{e^t-1}\mathrm{d}t$ on whole $\mathbb{C}$!
Now we can transform 1.1.7 to

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1}\mathrm{d}t.$$

As shown, the right-hand side has a continuation that is meromorphic on $\mathbb{C}$. So we also have this for $\zeta(s)$. Furthermore, the simple zeros of $\frac{1}{\Gamma}$ cancel with the simple (or removable) poles of the continuation of the integral in $s = 0, -1, -2, ...$ and become removable; it remains the only pole at $s = 1$. We get the following result:

Theorem 1.1.9. *The function $\zeta(s)$ has a meromorphic continuation to $\mathbb{C}$. Its only pole at $s = 1$ is simple.*

Remark. From now on we will now call $\zeta(s)$ the (unique) continuation of the zeta function.

Theorem 1.1.10. *For $s \in \mathbb{H}_1$ the following identity holds:*

$$\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}} \tag{1.1.10}$$

Proof. To increase readability we donate just once: p will be always prime and $s \in \mathbb{H}_1$ in this proof.
The key observation is that $\frac{1}{1-p^{-s}}$ can be written as a uniformly convergent geometric power series:

$$\frac{1}{1 - p^{-s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots .$$

Taking formally the product of these series over all primes p would yield the desired result. The precise proof goes like this:

Let M be a positive integer. We know by the fundamental theorem of arithmetic that any positive integer $n \leq M$ can be written uniquely up to order as a product of prime numbers. Each prime that occur in this product is less or equal than M and is repeated less than M times. Hence for $s = \sigma + it$ and $\sigma > 1$:

$$\begin{aligned}\left|\zeta(s) - \prod_{p \leq M} \frac{1}{1-p^{-s}}\right| &= \left|\zeta(s) - \prod_{p \leq M}\left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{Ms}}\right)\right| \\ &< \left|\sum_{n=M+1}^{\infty} \frac{1}{n^s}\right| \leq \sum_{n=M+1}^{\infty} \frac{1}{n^\sigma}.\end{aligned}$$

The first inequality is true because of the triangle inequality and the fact that each summand $\frac{1}{n^s}$ where $n \leq M$ will appear in the product and the second inequality is just the triangle inequality together with the property that $|n^s| = n^\sigma$. Letting M tend to infinity gives us the identity we wanted to show because:

$$\sum_{n=M+1}^{\infty} \frac{1}{n^\sigma} < \int_{M+1}^{\infty} \frac{1}{(t-1)^\sigma} \mathrm{d}t = \frac{1}{(\sigma-1)M^{\sigma-1}} \overset{M\to\infty}{\longrightarrow} 0.$$

□

This identity gives us a lot of information about the zeta function! For example we know that if an infinite product converges and vanishes in a point then one of its factors must equal to zero in this point (See [3] on page 141). It is easy to see that $\frac{1}{1-p^{-s}} \neq 0$ for any $s \in \mathbb{H}_1$ and any $p \in \mathbb{P}$. So the infinite product in our identity never equals to zero and so neither does $\zeta(s)$. We get the following important result:

Corollary 1.1.11. *Let $s \in \mathbb{H}_1$. Then:*

$$\zeta(s) \neq 0$$

It is not hard to show that $\zeta(s)$ has zeros at $s = -n, n \in \mathbb{N}$ iff the coefficient c_n of $f(z)$ in *Equation* (1.1.9) equals to zero. In fact this is the case for every even n. Moreover these coefficients are, by a simple corollary of the definition, closely connected to Bernoulli numbers! But as we do not need these proofs and exact values here they will not be provided in this work.
Also, it would be natural to ask "What about the other zeros?" and try to handle the remaining. The famous Riemann hypothesis states that all the other zeros of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$. This is and stays one of the most infamous unsolved problems in modern mathematics which would, if proven, solve many, many other conjectures. More information about this unsolved mystery can be

found in [1], [2] as well as a "friendly introduction" into this topic in [6].

Fortunately, for the proof of the Prime Number Theorem we only need the fact that:

Theorem 1.1.12. *$\zeta(s)$ has no zeros on the line* $\mathrm{Re}(s) = 1$

To prove this theorem we will need two lemmas: *Lemma* 1.1.14 and *Lemma* 1.1.15 and an important but very technical result in complex analysis which will not be proven here, since proving these theorems would go beyond the scope of this work. The detailed proof of the following theorem can be found in [3] on the pages 100-101 and as a exercise in [4] on page 62.

Theorem 1.1.13. *Let $\Omega \subset \mathbb{C}$ be a simply connected region and f a holomorphic function on Ω that has no zeros there. Then there exists a holomorphic function g such that:*

$$f(z) = e^{g(z)},$$

for $z \in \Omega$.

It can be easily shown that this function g is unique modulo $2\pi i$.
We know that $\zeta(s)$ has no zeros on $\mathbb{H}_1$ and so we can apply the theorem in this simply connected region. So we have deduced that there exists a function g such that for any $s \in \mathbb{H}_1$ holds:

$$\zeta(s) = e^{g(s)}.$$

The functions $g(s)$ we will naturally $\ln \zeta(s)$ and observe them modulo $2\pi i$ for uniqueness. Now we can formulate and prove the next lemma:

Lemma 1.1.14. *For $s \in \mathbb{H}_1$ it holds:*

$$\ln \zeta(s) = \sum_{\substack{p \in \mathbb{P} \\ m \in \mathbb{N}}} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} \frac{c_n}{n^s} \tag{1.1.11}$$

for some $c_n \geq 0$.

Proof. We will use the power series expansion for the logarithm for $0 \leq x < 1$:

$$\ln\left(\frac{1}{1-x}\right) = \sum_{m=1}^{\infty} \frac{x^m}{m},$$

which can be easily shown by differentiating both sides and using the formula for the geometric series. First we prove the lemma for $s \in \mathbb{R}_{>1}$ by taking the logarithm of the zeta functions product form:

$$\ln \zeta(s) = \ln \prod_{p \in \mathbb{P}} \frac{1}{1-p^{-s}} = \sum_{p \in \mathbb{P}} \ln\left(\frac{1}{1-p^{-s}}\right) = \sum_{p \in \mathbb{P}} \sum_{m \in \mathbb{N}} \frac{p^{-ms}}{m} = \sum_{\substack{p \in \mathbb{P} \\ m \in \mathbb{N}}} \frac{p^{-ms}}{m}.$$

As $s > 1$ implies that $0 < p^{-s} < 1$ we were allowed to use the series expansion of the logarithm and because the sum is converging absolutely we do not have to specify the summation order - this is a very technical result and is not proven in this work; nevertheless the reader can look into the proof in [3] on pages 197-199. Now by analytic continuation, the identity must hold on whole $\mathbb{H}_1$.
Finally putting $c_n := \frac{1}{m}$ if $n = p^m$ (p prime and $m \in \mathbb{N}$) and $c_n := 0$ otherwise proves the second equality part of the lemma. □

Now we will prove the second lemma that we need for *Theorem* 1.1.12. It might first look very strange but as we will see soon exactly this lemma is essential for the proof:

Lemma 1.1.15. *For $s \in \mathbb{H}_1$ ($s = \sigma + it$) it holds:*

$$\left|\frac{\zeta(\sigma+it)}{\sigma-1}\right|^4 |\zeta(\sigma+2it)||\zeta(\sigma)(\sigma-1)|^3 \geq \frac{1}{\sigma-1} \tag{1.1.12}$$

Proof. Of course *Equation* (1.1.12) is equivalent to

$$|\zeta^4(\sigma+it)||\zeta(\sigma+2it)||\zeta^3(\sigma)| \geq 1,$$

which is, after taking the logarithm, the same as:

$$4\ln|\zeta(\sigma+it)| + \ln|\zeta(\sigma+2it)| + 3\ln|\zeta(\sigma)| \geq 0.$$

Now we know that for any $z \in \mathbb{C}$ it holds that $\ln|z| = \mathrm{Re}(\ln z)$ and so it remains to prove that:

$$4\,\mathrm{Re}\big(\ln\zeta(\sigma+it)\big) + \mathrm{Re}\big(\ln\zeta(\sigma+2it)\big) + 3\,\mathrm{Re}\big(\ln\zeta(\sigma)\big) \geq 0.$$

Finally note that:

$$\begin{aligned}\mathrm{Re}(n^{-z}) &= \mathrm{Re}(e^{(-\sigma-it)\ln n}) \\ &= \mathrm{Re}\Big(e^{-\sigma\ln n}\big(\cos(t\ln n) + i\sin(-t\ln n)\big)\Big) \\ &= n^{-\sigma}\cos(t\ln n)\end{aligned}$$

and so we have for the same c_n as in *Lemma* 1.1.14:

$$\begin{aligned}
&4\operatorname{Re}\big(\ln\zeta(\sigma+it)\big)+\operatorname{Re}\big(\ln\zeta(\sigma+2it)\big)+3\operatorname{Re}\big(\ln\zeta(\sigma)\big)\\
&\qquad=4\operatorname{Re}\left(\sum_{n=1}^{\infty}\frac{c_n}{n^{\sigma+it}}\right)+\operatorname{Re}\left(\sum_{n=1}^{\infty}\frac{c_n}{n^{\sigma+2it}}\right)+3\operatorname{Re}\left(\sum_{n=1}^{\infty}\frac{c_n}{n^{\sigma}}\right)\\
&\qquad=4\left(\sum_{n=1}^{\infty}\frac{c_n\cos(t\ln n)}{n^{\sigma}}\right)+\left(\sum_{n=1}^{\infty}\frac{c_n\cos(2t\ln n)}{n^{\sigma}}\right)+3\left(\sum_{n=1}^{\infty}\frac{c_n}{n^{\sigma}}\right)\\
&\qquad=\sum_{n=1}^{\infty}\frac{c_n}{n^{\sigma}}(3+4\cos\theta_n+\cos 2\theta_n)=\sum_{n=1}^{\infty}\frac{c_n}{n^{\sigma}}2(1+\cos\theta_n)^2
\end{aligned}$$

where $\theta_n := t\ln n$. The last sum is obviously positive since every summand is not negative. $\square$

Now we can prove our *Theorem* 1.1.12:

Proof. Suppose $\operatorname{Re}(s_0)=1$ and $\zeta(s_0)$ vanishes. We can write $s=1+it_0$ for some $t_0\in\mathbb{R}^*$. Since $\zeta(s)$ is holomorphic on $\mathbb{C}\setminus\{1\}$ we must have $\lim_{\sigma\searrow 1}\zeta(\sigma+it_0)=\zeta(1+it_0)=0$. The left-hand side of *Equation* (1.1.12) for $t=t_0$ and $\sigma\searrow 1$ has then a finite limit:

$$\lim_{\sigma\searrow 1}\left(\left|\frac{\zeta(\sigma+it_0)}{\sigma-1}\right|^4|\zeta(\sigma+2it_0)||\zeta(\sigma)(\sigma-1)|^3\right)=|\zeta'(1+it_0)|^4|\zeta(1+2it_0)|,$$

because $\zeta(s)$ has a simple pole at $s=1$. But the right-hand side $\left(\frac{1}{\sigma-1}\right)$ goes to infinity as $\sigma\searrow 1$! This is a contradiction and so $\zeta(s)\neq 0$ whenever $\operatorname{Re}(s)=1$ $\square$

The last property of our zeta function that we need for the Prime Number Theorem is an inequality (or boundary) for its logarithmic derivative. In particular:

Theorem 1.1.16. *Let $s=\sigma+it$ and $\sigma\geq 1$, $|t|\geq 1$. Then for any $\eta>0$ there exists a constant A such that:*

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right|\leq A|t|^{\eta} \tag{1.1.13}$$

M. Stein and Rami Shakarchi call this theorem "a quantitative version of" *Theorem* 1.1.12 in their book "Complex Analysis" - in [3] on page 187. The reader can find the proof of this theorem there whereby that this particular boundary is first shown for the derivative of ζ - on pages 173 and 174 - and then for $\frac{1}{\zeta}$ - on pages 187 and 188. Of course multiplying these inequalities together justifies *Theorem* 1.1.16.

Now we have enough properties of the zeta function to tackle the Prime Number Theorem.

Chapter 2

The Prime Number Theorem

The goal of this chapter is to prove the Prime Number Theorem with the help of the zeta function. Nevertheless we start with some interesting historic information about it the theorem:

2.1 Historic notes and definitions

The reader hopefully remembers from the introduction that the aim of this thesis is to prove the Prime Number Theorem and that it is about estimating the amount of primes under a given value. In other words estimating the growth of the function we now introduce:

Definition. For $x \in \mathbb{R}^+$ we define: $\pi(x) := \#\{p \in \mathbb{P} : p \leq x\}$.

As we already started formal definitions now it is time to write the statement of the Prime Number Theorem down. To do so we need another definition that will help us talk about "estimating growth":

Definition. We call two functions $f(x)$ and $g(x)$ **asymptotically equal** and write $f \sim g$ iff:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

Now we can finally state the Prime Number Theorem:

Theorem 2.1.1. *(The Prime Number Theorem)*

$$\pi(x) \sim \frac{x}{\ln x}. \tag{2.1.1}$$

Arguably first, after studying tables of prime numbers, Adrien-Marie Legendre conjectured in 1797 that this function is approximately $x/(A \ln x + B)$ for some constants A and B. About 10 years later he specified A being 1. A lot of work in this field has been done after this conjecture by Carl Friedrich Gauss and Peter Gustav Lejeune Dirichlet but it lasted almost half a century until the Russian mathematician Pafnuty Chebyshev made an important step forward in this field using the already introduced in this work zeta function, but only for real inputs s. He proved that the limit of $\frac{\pi(x)}{x/\ln x}$ as x goes to infinity exists.
The real breakthrough regarding the Prime Number Theorem was made by Bernhard Riemann in his only work on this topic "On the Number of Primes Less Than a Given Magnitude". It can be found translated in English in [1]. There Riemann introduced the idea to tackle ζ with complex analysis methods as it was described in the previous chapter of this thesis. Using these ideas Jacques Hadamard and Charles Jean de la Vallée-Poussin independently proved the Prime Number Theorem in the same year of 1896.
During the last century, these proofs have been simplified and many other ones were discovered including one "elementary" proof by Atle Selberg and Paul Erdős and a relatively short and ingenious one by Donald J. Newman.

2.2 Definitions and properties of important functions

To successfully deal with the Prime Number Theorem we need some more useful functions from number theory. We start with formal definitions:

Definition. The von Mangoldt function is defined as:

$$\Lambda(n) := \begin{cases} \ln n, & \text{when } n = p^k\text{, where k is an integer and p is prime} \\ 0, & \text{otherwise.} \end{cases}$$

Definition. The Tchebychev ψ-function is defined as:

$$\psi(x) := \sum_{\substack{p^m \leq x \\ p \in \mathbb{P}, m \in \mathbb{N}}} \ln p = \sum_{n \leq x} \Lambda(n)$$

.

It turns out that it is easier to work with a slightly modified version of the Tchebychev ψ-function, namely:

Definition. We define the ψ_1 as:

$$\psi_1(x) := \int_0^x \psi(t)\mathrm{d}t.$$

We saw that the statement of the Prime Number Theorem is closely connected to the term of asymptotic behavior. To handle it we have a more useful criterion then the definition: To prove that $f \sim g$ it suffices to prove that:

$$1 \leq \liminf_{x\to\infty} \frac{f(x)}{g(x)} \qquad \text{and} \qquad \limsup_{x\to\infty} \frac{f(x)}{g(x)} \leq 1.$$

This follows immediately from the fact that:

$$\liminf_{x\to\infty} \frac{f(x)}{g(x)} \leq \lim_{x\to\infty} \frac{f(x)}{g(x)} \leq \limsup_{x\to\infty} \frac{f(x)}{g(x)}.$$

All the useful functions defined, we can now start proving the relations between them. As we will see in the next proposition, it turns out that the functions ψ_1,ψ and π are closely connected to each other regarding asymptotic behavior:

Proposition 2.2.1. *The following relations are true:*

$$\psi_1(x) \sim \frac{x^2}{2} \Rightarrow \psi(x) \sim x \Rightarrow \pi(x) \sim \frac{x}{\ln x}.$$

Proof. For the first implication note that $\psi(x)$ is obviously increasing, so for any $\alpha < 1 < \beta$ we have:

$$\frac{1}{(1-\alpha)x}\int_{\alpha x}^{x} \psi(u)\mathrm{d}u \leq \psi(x) \leq \frac{1}{(\beta-1)x}\int_{x}^{\beta x} \psi(u)\mathrm{d}u. \tag{2.2.1}$$

The second inequality implies that

$$\psi(x) \leq \frac{1}{(\beta-1)x}\big(\psi_1(\beta x) - \psi_1(x)\big),$$

which is equivalent to

$$\frac{\psi(x)}{x} \leq \frac{1}{(\beta-1)}\left(\frac{\psi_1(\beta x)}{(\beta x)^2}\beta^2 - \frac{\psi_1(x)}{x^2}\right).$$

And, since we assume that $\lim\limits_{x\to\infty} \frac{\psi_1(x)}{x^2} = \frac{1}{2}$, we have:

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} \leq \frac{1}{(\beta-1)}\left(\lim_{x\to\infty} \frac{\psi_1(\beta x)}{(\beta x)^2}\beta^2 - \lim_{x\to\infty} \frac{\psi_1(x)}{x^2}\right) = \frac{\beta^2-1}{2(\beta-1)} = \frac{\beta+1}{2}.$$

As this is true for any $\beta > 1$, we have proved that $\limsup\limits_{x\to\infty} \frac{\psi(x)}{x} \leq 1$.
The second inequality similarly gives us:

$$\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq \frac{1-\alpha^2}{2(1-\alpha)} = \frac{\alpha+1}{2},$$

which is again true for any $\alpha < 1$ and so we have here $\liminf\limits_{x\to\infty} \frac{\psi(x)}{x} \geq 1$.
So we have

$$\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq 1 \geq \limsup_{x\to\infty} \frac{\psi(x)}{x},$$

which proves the first implication in *Proposition* 2.2.1.

For the second implication we note that:

$$\psi(x) = \sum_{\substack{p^m \leq x \\ p\in\mathbb{P}, m\in\mathbb{N}}} \ln p = \sum_{\substack{p\leq x \\ p\in\mathbb{P}}} \left\lfloor \frac{\ln x}{\ln p} \right\rfloor \ln p.$$

That this is true one can easily see by counting how much the summand $\ln p$ for a fixed p appears in each sum. On the right-hand side it is obviously exactly $\left\lfloor \frac{\ln x}{\ln p} \right\rfloor$ times. And left we have, since $p^m \leq x$ is equivalent to $m \leq \ln x / \ln p$, that $\ln p$ appears exactly once for each term in the set $\{p, p^2, p^3, ..., p^{\lfloor \ln x / \ln p \rfloor}\}$, so also exactly $\left\lfloor \frac{\ln x}{\ln p} \right\rfloor$ times. This gives us:

$$\psi(x) = \sum_{\substack{p\leq x \\ p\in\mathbb{P}}} \left\lfloor \frac{\ln x}{\ln p} \right\rfloor \ln p \leq \sum_{\substack{p\leq x \\ p\in\mathbb{P}}} \frac{\ln x}{\cancel{\ln p}} \cancel{\ln p} = \pi(x) \ln x,$$

and dividing through by x yields:

$$\frac{\psi(x)}{x} \leq \frac{\pi(x)\ln x}{x}.$$

Since we assume here that $\psi(x) \sim x$ we have:

$$1 \leq \liminf_{x\to\infty} \pi(x) \frac{\ln x}{x}. \tag{2.2.2}$$

The other inequality is a little bit trickier. To prove it we first fix $0 < \alpha < 1$ and assess:

$$\psi(x) = \sum_{\substack{p^m \le x \\ p \in \mathbb{P}, m \in \mathbb{N}}} \ln p \ge \sum_{\substack{p \le x \\ p \in \mathbb{P}}} \ln p \ge \sum_{\substack{x^\alpha < p \le x \\ p \in \mathbb{P}}} \ln p$$
$$\ge \sum_{\substack{x^\alpha < p \le x \\ p \in \mathbb{P}}} \ln x^\alpha = \big(\pi(x) - \pi(x^\alpha)\big)\alpha \ln x.$$

This implies after again dividing through by x:

$$\frac{\psi(x)}{x} + \frac{\alpha\pi(x^\alpha)\ln x}{x} \ge \frac{\alpha\pi(x)\ln x}{x}. \tag{2.2.3}$$

Now, because $\alpha < 1$ and $\pi(x^\alpha) \le x^\alpha$, we have

$$\frac{\alpha\pi(x^\alpha)\ln x}{x} \le \alpha\frac{\ln x}{x^{1-\alpha}} \xrightarrow{x\to\infty} 0,$$

and so *Equation* (2.2.3) implies that:

$$1 \ge \alpha \limsup_{x\to\infty} \pi(x)\frac{\ln x}{x}, \tag{2.2.4}$$

for any arbitrary $\alpha < 1$. The proof is complete after comparing *Equation* (2.2.2) and *Equation* (2.2.4). □

Remark. '⇒' in Proposition 2.2.1 can actually be replaced by '⇔' but since we do not need this results we leave it as an exercise for the interested reader. See [3].

Proposition 2.2.1 implies that to prove the Prime Number Theorem (see *Theorem* 2.1.1) we have to justify:

$$\psi_1(x) \overset{!}{\sim} \frac{x^2}{2}.$$

This brings us to the next section:

2.3 Connection to the zeta function

As we already saw, the key to the Prime Number Theorem is understanding the behavior of our ψ_1-Function. Exactly here the zeta function joins by the following theorem:

Theorem 2.3.1. *We define:*

$$\mathcal{F}(s) := \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right). \tag{2.3.1}$$

Then for $c \in \mathbb{R}_{>1}$ *it holds:*

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{F}(s) \mathrm{d}s. \tag{2.3.2}$$

Proving this theorem is the goal of this section, but already now the reader hopefully sees that fighting with the zeta function and extracting properties from it was very helpful as it is directly connected to ψ_1 and so to the Prime Number Theorem.
We start with some lemmas:

Lemma 2.3.2. *The following identity is true:*

$$\psi_1(x) = \sum_{n \leq x} \Lambda(n)(n - x). \tag{2.3.3}$$

Proof. By definition we have:

$$\psi(u) = \sum_{n \leq x} \Lambda(n) = \sum_{n=1}^{\infty} \Lambda(n) f_n(u),$$

where $f_n(u) := 1$ if $n \leq u$ and $f_n(u) = 0$ otherwise. Therefore, we see:

$$\begin{aligned}
\psi_1(x) &= \int_0^x \psi(u) \mathrm{d}u \\
&= \int_0^x \sum_{n=1}^{\infty} \Lambda(n) f_n(u) \mathrm{d}u = \sum_{n=1}^{\infty} \int_0^x \Lambda(n) f_n(u) \mathrm{d}u \\
&= \sum_{n=1}^{\infty} \Lambda(n) \int_0^x f_n(u) \mathrm{d}u = \sum_{n \leq x} \Lambda(n) \left(\int_0^n f_n(u) \mathrm{d}u + \int_n^x f_n(u) \mathrm{d}u \right) \\
&= \sum_{n \leq x} \Lambda(n) \left(\int_0^n 0 \mathrm{d}u + \int_n^x 1 \mathrm{d}u \right) = \sum_{n \leq x} \Lambda(n)(n - x).
\end{aligned}$$

□

The second lemma is now not hard to prove, as we have worked through so many theory of the zeta function:

Lemma 2.3.3. *For* $s \in \mathbb{H}_1$ *it holds:*

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}. \tag{2.3.4}$$

Proof. First observe that, since $\Lambda(n) = 0$ whenever $n \neq p^m$ for some m integer and a p prime, we can rewrite the right-hand side:

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_{\substack{p\in\mathbb{P}\\ m\in\mathbb{N}}} \frac{\Lambda(p^m)}{p^{ms}} = \sum_{\substack{p\in\mathbb{P}\\ m\in\mathbb{N}}} \frac{\ln p}{p^{ms}}.$$

Note that we do not have to specify the order of summation again, since the sums over $p \in \mathbb{P}$ and over $m \in \mathbb{N}$ converge absolutely.
Now we use the formula from *Lemma* 1.1.14 that on $s \in \mathbb{H}_1$:

$$\ln \zeta(s) = \sum_{\substack{p\in\mathbb{P}\\ m\in\mathbb{N}}} \frac{p^{-ms}}{m},$$

and differentiate both sides to get:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{\substack{p\in\mathbb{P}\\ m\in\mathbb{N}}} (\ln p)p^{-ms} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Finally multiplying both sides with -1 yields the desired result. $\square$

The last lemma we need for *Theorem* 2.3.1 is a computation of a contour integral:

Lemma 2.3.4. *For $c \in \mathbb{R}_{>0}$ and $a \in \mathbb{R}^+$ it holds:*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(s+1)}\mathrm{d}s = \begin{cases} 0 & \text{if } 0 < a \leq 1 \\ 1 - \frac{1}{a} & \text{if } 1 < a. \end{cases} \tag{2.3.5}$$

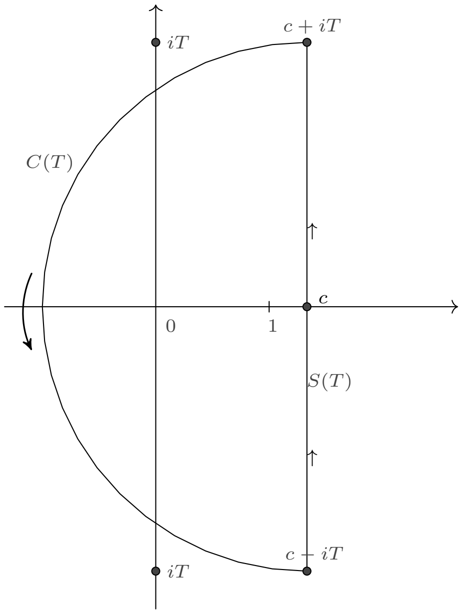
This notation means here that we are integrating "upwards" over the vertical line $\mathrm{Re}(s) = c$*.*

Proof. Note that since over this contour of integration $|a^s| = a^c$ and so the integral converges and is well-defined.

In the case $1 < a$ we can rewrite $a = e^{\beta}$ and $\beta > 0$ and we can define:

$$f(s) := \frac{a^s}{s(s+1)} = \frac{e^{s\beta}}{s(s+1)}$$

After simple computation we get $\mathrm{Res}_{s=0} f(s) = 1$ and $\mathrm{Res}_{s=-1} f(s) = -\frac{1}{a}$. Now to compute the integral, consider for $T > 0$ the closed path $\Gamma(T)$ like in figure below:



The path consists of the straight vertical segment $S(T)$ from the point $c - iT$ to $c + iT$ and from there the halfcircle $C(T)$ with the center at c and radius T, lying to the left of the vertical segment and so going back to $c - iT$. Also let $\Gamma(T)$ be positive oriented.
Now choose T large enough that 0 and 1 are both in the interior of $\Gamma(T)$. Then by the residue formula we have:

$$\frac{1}{2\pi i}\int_{\Gamma(T)} f(s)\mathrm{d}s = 1 - \frac{1}{a}.$$

Since we also can write the integral as follows:

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma(T)} f(s)\mathrm{d}s &= \frac{1}{2\pi i}\int_{S(T)} f(s)\mathrm{d}s + \frac{1}{2\pi i}\int_{C(T)} f(s)\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{a^s}{s(s+1)}\mathrm{d}s + \frac{1}{2\pi i}\int_{C(T)} f(s)\mathrm{d}s,\end{aligned}$$

it remains to prove that the integral over $C(T)$, so over the halfcircle, goes to 0 as $T \to \infty$. To show this note that for T large enough and $s = \sigma + it \in C(T)$ we have $|s| > T/2$ and $|s+1| > T/2$, which gives us:

$$\left|\frac{1}{s(s+1)}\right| = \frac{1}{|s||s+1|} < \frac{4}{T^2},$$

and since $\sigma \leq c$ and β was positive we also have $|e^{\beta s}| = e^{\beta\sigma} \leq e^{\beta c}$. Hence, because the length of the halfcircle $C(T)$ is πT we get:

$$\left|\int_{C(T)} f(s)\mathrm{d}s\right| < \frac{4}{T^2}e^{\beta c}\cdot \pi T \overset{T\to\infty}{\longrightarrow} 0,$$

which finishes the case $1 < a$.

If now $0 < a \leq 1$ consider the analogous contour but with the halfcircle lying to the right of the line $\mathrm{Re}(s) = c$. We will call them $\Gamma'(T)$, $S(T)$ and $C'(T)$ analogously. Then $f(s)$ has no poles in the interior of $\Gamma'(T)$ and by the same argumentation as in the other case we will have to show here that the integral over $C'(T)$ tends to zero as $T \to \infty$. Moreover, the same inequalities hold for s: $|s| > T/2$ and $|s+1| > T/2$ for large enough T. The difference is at β, because, since $0 < a \leq 1$, we have that $a^s = e^{\beta s}$, where $\beta \leq 0$. For $s = \sigma + it \in C'(t)$ we have of course that $\sigma \geq c$ and because $\beta \leq 0$ we get $\sigma\beta \leq c\beta$. Hence, $|e^{\beta s}| = e^{\beta\sigma} \leq e^{c\beta}$. Now we can assess again to get similarly to the other case:

$$\left|\int_{C'(T)} f(s)\mathrm{d}s\right| < \frac{4}{T^2}e^{\beta c}\cdot \pi T \overset{T\to\infty}{\longrightarrow} 0$$

The proof is complete. $\square$

Now we are finally ready to prove *Theorem* 2.3.1:

Proof. Lemma 2.3.3 implies that:

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\mathrm{d}s &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right)\mathrm{d}s \\ &= \sum_{n=1}^{\infty}\Lambda(n)x\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(x/n)^s}{s(s+1)}\mathrm{d}s.\end{aligned}$$

Now comes *Lemma* 2.3.4 into play. For every $n < x$ we have $x/n > 1$ and the integral gives us $1 - \frac{n}{x}$ and for all the other n we get $x/n \leq 1$ and the integral is equal to 0. In other words we have:

$$\sum_{n=1}^{\infty}\Lambda(n)x\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(x/n)^s}{s(s+1)}\mathrm{d}s = \sum_{n\leq x}\Lambda(n)x\big(1-\frac{n}{x}\big) = \sum_{n\leq x}\Lambda(n)(x-n).$$

But this is exactly $\psi_1(x)$ as *Lemma* 2.3.2 states:

$$\sum_{n\leq x}\Lambda(n)(x-n) = \psi_1(x),$$

and so the proof of *Theorem* 2.3.1 is complete. $\square$

With this finished we have now all the ingredients to move on to the next section:

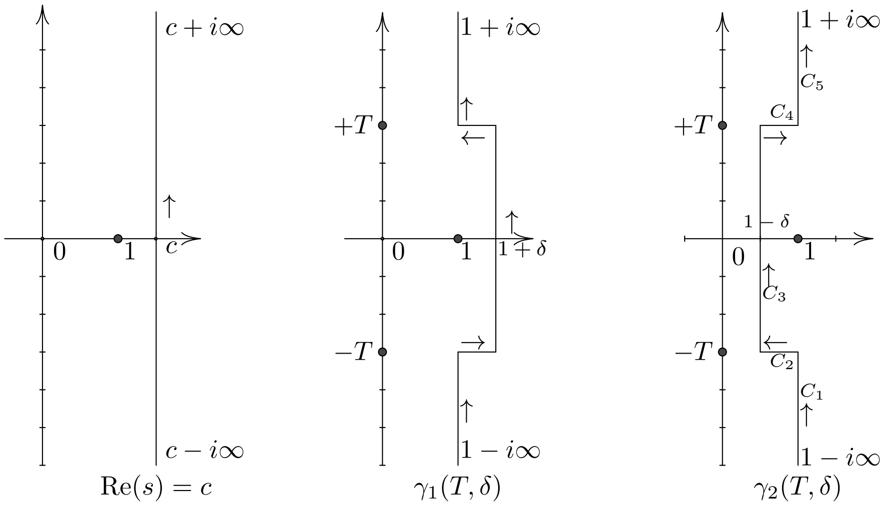
2.4 Proof of the Prime Number Theorem

In this section we will show that the integral of $\mathcal{F}(s)$ that describes the ψ_1-function in Theorem 2.3.1 is asymptotically equal to $x^2/2$. This, together with Theorem 2.3.1 and Proposition 2.2.1 proves the Prime Number Theorem. This proof relies on the one in [3] on the pages 194-197.
To handle this integral we will have to to integrate $\mathcal{F}(s)$ not over the line $\mathrm{Re}(s) = c$ with $c > 1$ but on $\mathrm{Re}(s) = 1$. The problem that $\mathcal{F}(s)$ has a pole on $s = 1$, we will solve by integrating around it. Luckily this is the only pole on $\mathrm{Re}(s) = 1$ as we will see.

The basic strategy explained let us start with the detailed proof of the Prime Number Theorem:
For $T, \delta \in \mathbb{R}^+$ we claim:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{F}(s)\mathrm{d}s = \frac{1}{2\pi i}\int_{\gamma_1(T,\delta)} \mathcal{F}(s)\mathrm{d}s, \tag{2.4.1}$$

where $\gamma_1(T, \delta)$ is the path on $\mathbb{C}$ defined like in the Figure below:



We go on the line $\mathrm{Re}(s) = 1$ upwards until $\mathrm{Im}(s) = -T$, then to the point $s = 1 + \delta - Ti$, after that to $s = 1 + \delta + Ti$, then back to $\mathrm{Re}(s) = 1$ and finally again up, to infinity.
To justify this transformation we of course use Cauchy's Theorem, we just have to explain two facts: $\mathcal{F}(s)$ has no poles between $\gamma_1(T, \delta)$ and $\mathrm{Re}(s) = c$ and the

decrease of this function at infinity is rapid enough.
$\mathcal{F}(s)$ is obviously holomorphic on $\mathbb{H}_1$ since $\frac{x^{s+1}}{s(s+1)}$, ζ and its logarithmic derivative are. On the line $\mathrm{Re}(s) = 1$ our function has the only pole at $s = 1$ because as *Theorem* 1.1.12 states the zeta function has no zeros on this line. This means that $\mathcal{F}(s)$ is indeed regular between $\gamma_1(T, \delta)$ and $\mathrm{Re}(s) = c$.
Theorem 1.1.16 tells us that the logarithmic derivative of $\zeta(s)$ is bounded by $A|t|^\eta$ for a constant A for any fixed η (where $s = \sigma + it$, $\sigma \geq 1$ and $|t| \geq 1$). So $|\mathcal{F}(s)| \leq B|t|^{-2+\eta}$ on $\gamma_1(T, \delta)$ and between $\gamma_1(T, \delta)$ and $\mathrm{Re}(s) = c$ for a constant B. This guarantees the rapid enough decrease.

Now we want to pass this integral to the one over $\gamma_2(T, \delta)$, where we use the same path as in $\gamma_1(T, \delta)$, but going left around 1 and not right - see again the same Figure. In this transformation the Residue Theorem will help us: we choose (for a fixed T) δ small enough that $\zeta(s)$ has no zeros in the box

$$\{s = \sigma + it, 1 - \delta \leq \sigma \leq 1, |t| \leq T\},$$

which is possible because ζ does not vanish on the line $\mathrm{Re}(s) = 1$.
Now let $C(T, \delta)$ be the closed rectangle path that connects this points with straight lines in this order: $-iT, -iT + \delta, iT + \delta, iT - \delta, -iT - \delta, -iT$. Because between $\gamma_1(T, \delta)$ and $\gamma_2(T, \delta)$ except in $s = 1$ the function $\mathcal{F}(s)$ is regular it clearly holds by Cauchy's Theorem and the Residue Theorem:

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma_1(T,\delta)} \mathcal{F}(s) \mathrm{d}s &= \frac{1}{2\pi i} \int_{\gamma_2(T,\delta)} \mathcal{F}(s) \mathrm{d}s + \frac{1}{2\pi i} \int_{C(T,\delta)} \mathcal{F}(s) \mathrm{d}s \\ &= \frac{1}{2\pi i} \int_{\gamma_2(T,\delta)} \mathcal{F}(s) \mathrm{d}s + \mathrm{Res}_{s=1} \mathcal{F}(s). \end{aligned}$$

To compute $\mathrm{Res}_{s=1}\mathcal{F}(s)$ note that $\mathcal{F}(s) = \frac{x^{s+1}}{s(s+1)} \cdot \left(-\frac{\zeta'(s)}{\zeta(s)}\right) =: f(s) \cdot g(s)$, where $f(s)$ is holomorphic at $s = 1$ and $g(s)$ is the logarithmic derivative of the zeta function. So it holds because the zeta function has a simple pole at $s = 1$:

$$\mathrm{Res}_{s=1}\mathcal{F}(s) = -\frac{x^{s+1}}{s(s+1)}\bigg|_{s=1} \mathrm{Res}_{s=1}\left(\frac{\zeta'(s)}{\zeta(s)}\right) = -\frac{x^2}{2} \cdot (-1) = \frac{x^2}{2}.$$

Our goal is now to assess the integral over $\gamma_2(T, \delta)$ of $\mathcal{F}(s)$. To do so we decompose $\gamma_2(T, \delta)$ as $C_1 + C_2 + C_3 + C_4 + C_5$, where each C_j is a straight line of $\gamma_2(T, \delta)$ which we numerate bottom-up like in the Figure above. Obviously each C_j depends on T and δ; so it would be correct to write $C_j^{T,\delta}$ but to increase readability we will just note this fact once here and continue writing C_j.

We deal with C_1 and C_5 first and show that after choosing an arbitrary small

$\varepsilon > 0$ there exists T so large that:

$$\left|\int_{C_1} \mathcal{F}(s)\mathrm{d}s\right| \leq \frac{\varepsilon}{2}x^2 \qquad \text{and} \qquad \left|\int_{C_5} \mathcal{F}(s)\mathrm{d}s\right| \leq \frac{\varepsilon}{2}x^2.$$

Let $j \in \{1,5\}$. First we note that for $s = \sigma + it \in C_j$ we have that $\sigma = 1$ and so $|s| > |t|$, $|s+1| > |t|$ and $|x^{1+s}| = x^{1+\sigma} = x^2$. By *Theorem* 1.1.16 we have for example for $\eta = 1/2$ that $\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leq A|t|^{1/2}$, which yields:

$$\left|\int_{C_j} \mathcal{F}(s)\mathrm{d}s\right| \leq \int_T^\infty \frac{x^2}{|t|^2}A|t|^{1/2}\mathrm{d}t = Ax^2\int_T^\infty t^{-3/2}\mathrm{d}t.$$

Since this integral converges, we can choose T so large that the integral is smaller then $\frac{\varepsilon}{2A}$ and then the right-hand side is indeed smaller then $\frac{\varepsilon}{2}x^2$.
On the last vertical C_3 we have $|x^{1+s}| = x^{1+1-\delta} = x^{2-\delta}$ and so we can conclude that $\mathcal{F}(s)$ is bounded by $M_{T,\delta}x^{2-\delta}$ on C_3 for some constant $M_{T,\delta} > 0$ that depends on T and δ. This implies that since the length of C_3 is of course $2T$:

$$\left|\int_{C_3} \mathcal{F}(s)\mathrm{d}s\right| \leq 2TM_{T,\delta}x^{2-\delta} =: M'_{T,\delta}x^{2-\delta},$$

for a constant M'_T that depends on T.

Finally, to deal with the horizontal segments C_2 and C_4 we can similarly conclude that $\mathcal{F}(s)$ is bounded by $K_{T,\delta}x^{s+1}$ ($K_{T,\delta} > 0$ again a constant depending on T and δ) on these segments. Let $j \in \{2,4\}$, then for $s = \sigma + it \in C_j$ we know hat σ goes from $1-\delta$ to 1 (or vice versa) and so we estimate for $x > 1$:

$$\left|\int_{C_j} \mathcal{F}(s)\mathrm{d}s\right| \leq K_{T,\delta}\int_{1-\delta}^1 x^{1+\sigma}\mathrm{d}\sigma = K_{T,\delta}\frac{x^2}{\ln x} - K_{T,\delta}\frac{x^{2-\delta}}{\ln x} < K_{T,\delta}\frac{x^2}{\ln x}.$$

In conclusion we have that there exist constants $K_{T,\delta}$ and $M'_{T,\delta}$ such that:

$$\begin{aligned}\left|\psi_1(x) - \frac{x^2}{2}\right| &= \left|\frac{1}{2\pi i}\int_{\gamma_1(T,\delta)} \mathcal{F}(s)\mathrm{d}s - \frac{x^2}{2}\right| \\ &= \left|\frac{1}{2\pi i}\int_{\gamma_2(T,\delta)} \mathcal{F}(s)\mathrm{d}s\right| \leq \left|\sum_{j=1}^5\int_{C_j} \mathcal{F}(s)\mathrm{d}s\right| \leq \sum_{j=1}^5\left|\int_{C_j} \mathcal{F}(s)\mathrm{d}s\right| \\ &\leq \varepsilon x^2 + M'_{T,\delta}x^{2-\delta} + 2K_{T,\delta}\frac{x^2}{\ln x}.\end{aligned}$$

Finally dividing through by $x^2/2$ gives us:

$$\left|\frac{2\psi_1(x)}{x^2} - 1\right| \leq 2\varepsilon + 2M'_{T,\delta}x^{-\delta} + 4K_{T,\delta}\frac{1}{\ln x}.$$

Now choose x so large that $2M'_{T,\delta}x^{-\delta} < \varepsilon$ and $4K_{T,\delta}\frac{1}{\ln x} < \varepsilon$. Then the left-hand

side is smaller than 4ε for any arbitrarily small $\varepsilon > 0$!
This implies that:

$$\psi_1(x) \sim \frac{x^2}{2},$$

and the proof of the Prime Number Theorem is complete.

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