Solving Rupert's Problem Algorithmically

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Rupert's Property

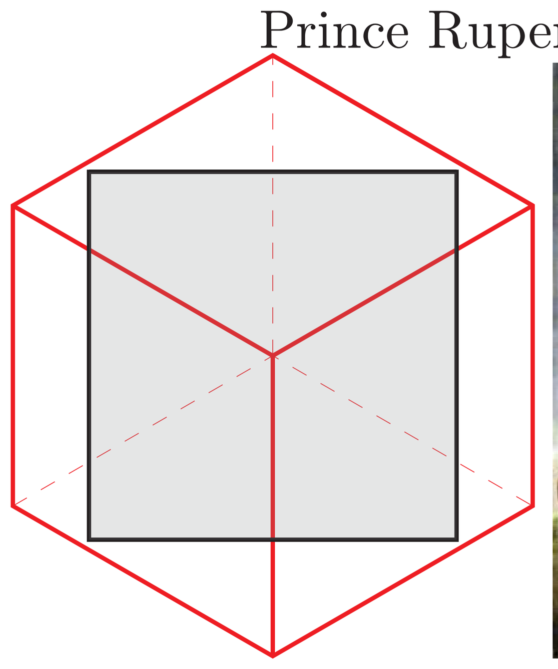
A convex polyhedron $\mathbf{P} \subset \mathbb{R}^3$ is *Rupert* (or has *Rupert's property*) if a hole (with the shape of a straight tunnel) can be cut into it such that a copy of $\mathbf{P}$ can be moved through this hole.

Rupert's problem is the task to decide whether a given polyhedron has Rupert's property.

The cube has Rupert's property

It is possible to cut a hole in the unit cube such that another unit cube can pass through it.

Prince Rupert & J. Wallis, 17th c.



Open Questions

- Are all convex polyhedra Rupert? [1, 4]
- Optimal solutions to Rupert's problem?
- Connection to dual solids?

Equivalent reformulation

A polyhedron $\mathbf{P}$ is Rupert if and only if there exist two projections $M_{\theta_1,\varphi_1}, M_{\theta_2,\varphi_2} : \mathbb{R}^3 \to \mathbb{R}^2$, a rotation $R_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ and a translation map $T_{x,y} : \mathbb{R}^2 \to \mathbb{R}^2$ such that the *polygon* $\mathcal{P} = (T_{x,y} \circ R_\alpha \circ M_{\theta_1,\varphi_1})(\mathbf{P})$ lies strictly inside the *polygon* $\mathcal{Q} = M_{\theta_2,\varphi_2}(\mathbf{P})$.

Probabilistic algorithm

Input: A polyhedron $\mathbf{P} = \{P_1, \dots, P_n\} \subseteq \mathbb{R}^3$.

Output: A solution $(x, y, \alpha, \theta_1, \theta_2, \varphi_1, \varphi_2) \in \mathbb{R}^7$ if $\mathbf{P}$ is Rupert.

(1) For each $i \in \{1, 2\}$, draw θ_i uniformly in $[0, 2\pi)$, and $\widetilde{\varphi}_i$ uniformly in $[-1, 1]$. Set $\varphi_i := \arccos(\widetilde{\varphi}_i)$.

(2) Construct the two 3×2 matrices A and B corresponding to the linear maps M_{θ_1,φ_1} and M_{θ_2,φ_2}. Compute the two projections of $\mathbf{P}$ given by $\mathcal{P}' := A \cdot \mathbf{P}$ and $\mathcal{Q}' := B \cdot \mathbf{P}$.

(3) Find the vertices on the convex hulls of $\mathcal{P}'$ and $\mathcal{Q}'$, and denote them by $\mathcal{P}$ and $\mathcal{Q}$.

(4) Decide whether $\mathcal{P}$ fits inside $\mathcal{Q}$, e.g. by using the algorithm from [2].

(5) If Step (4) yields a solution (x, y, α), return $(x, y, \alpha, \theta_1, \theta_2, \varphi_1, \varphi_2)$. Otherwise, repeat Steps (1)–(5).

Deterministic algorithm

Input: A polyhedron $\mathbf{P} = \{P_1, \dots, P_n\} \subseteq \mathbb{Z}^3$.

Output: A solution $(x, y, \alpha, \theta_1, \theta_2, \varphi_1, \varphi_2) \in \mathbb{R}^7$ if $\mathbf{P}$ is Rupert.

For every possible silhouette $s = (s_1, \dots, s_k)$ of $\mathbf{P}$ do:

(1) Define the system of inequalities $\det(Q_{s_{i+1}} - P_j, Q_{s_i} - P_j) > 0$ for $j = 1, \dots, n$ and $i = 1, \dots, k$, where $Q_i := M_{\theta_2,\varphi_2}(\mathbf{P}_{s_i})$ and $P_j := (T_{x,y} \circ R_\alpha \circ M_{\theta_1,\varphi_1})(\mathbf{P}_j)$ as well as $Q_{j+1} := Q_1$.

(2) Substitute the variables $\alpha, \theta_i, \varphi_i$ by a, b_i, c_i, using the parametrization of the circle. This yields a *system of rational inequalities*.

(3) Multiply each inequality by $\big((1+a^2)(1+b_1^2)(1+b_2^2)(1+c_1^2)(1+c_2^2)\big)^2$, to get a *system of polynomial inequalities* with integer coefficients.

(4) Search for a solution, e.g. by using the algorithm described in [3].

(5) If (4) yielded a solution: Transform back to the original variables $(x, y, \alpha, \theta_1, \theta_2, \varphi_1, \varphi_2)$ and return the solution.

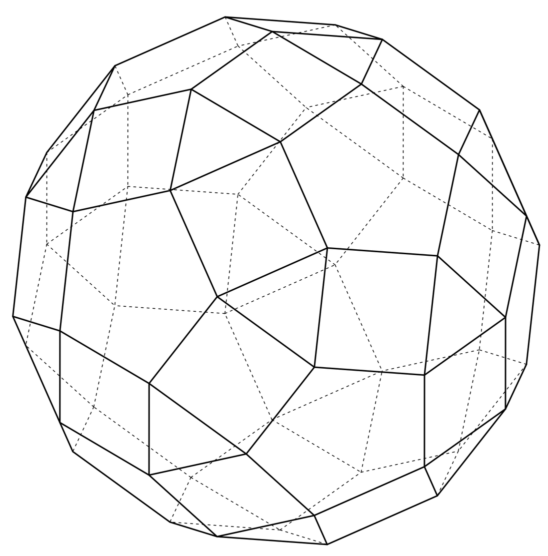
Theorem I

- All 5 Platonic solids are Rupert [4].
- At least 10 of 13 Archimedean solids have Rupert's property [1, 5, 6].
- At least 9 of 13 Catalan solids have Rupert's property [6].
- At least 82 of 92 Johnson solids have Rupert's property [6].

Conjecture

The rhombicosidodecahedron is not Rupert [6].

Remains to prove emptiness of 50 semi-algebraic sets each defined by 3600 polynomial inequalities in 6 variables of total degree 22.

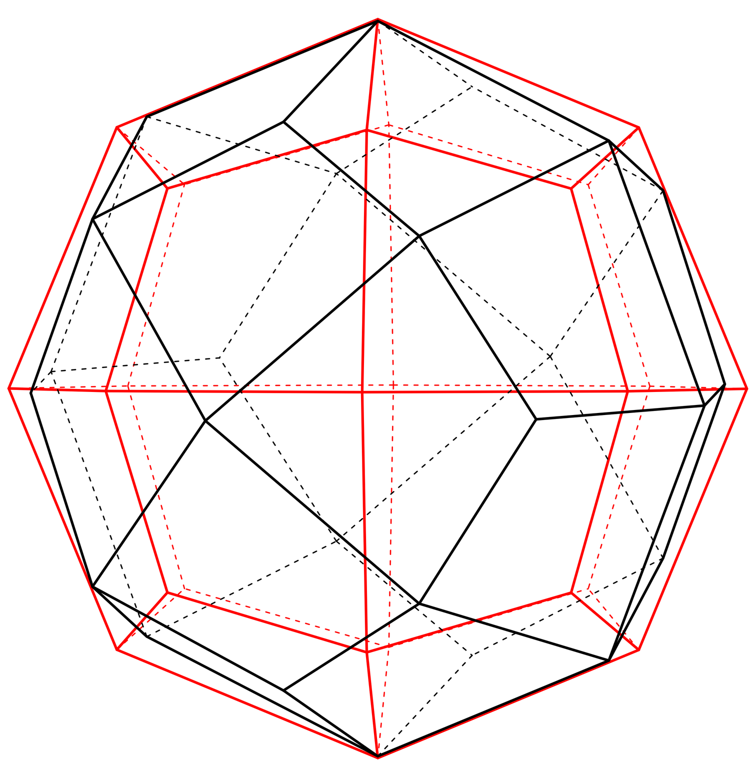


Theorem II

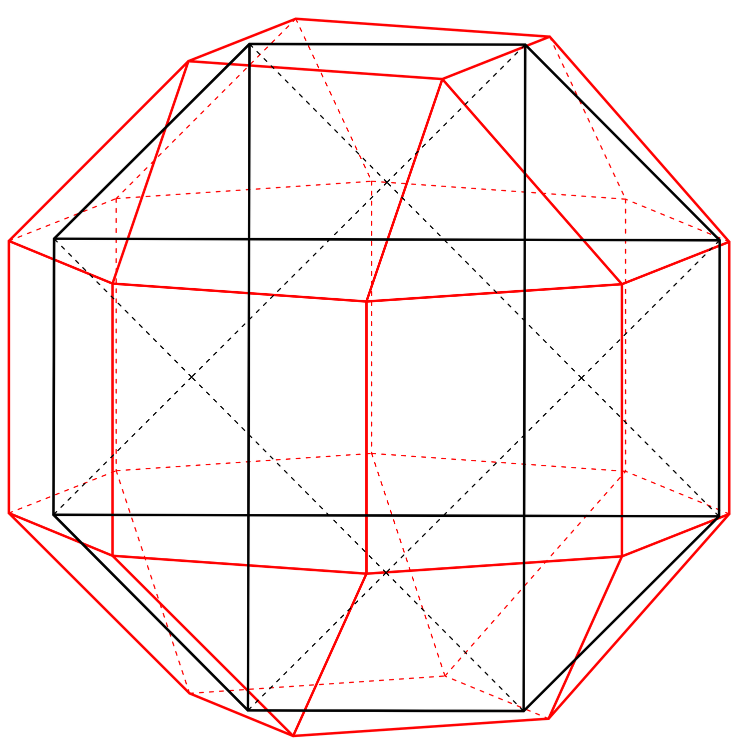
Rupert's problem is algorithmically decidable if $\mathbf{P}$ has algebraic coordinates. If $\mathbf{P}$ has rational coordinates, bounded in absolute value by m, then the running time of this algorithm is in $(\log(m) \cdot n)^{O(1)} \cdot n!$ [6].

Solutions for selected solids

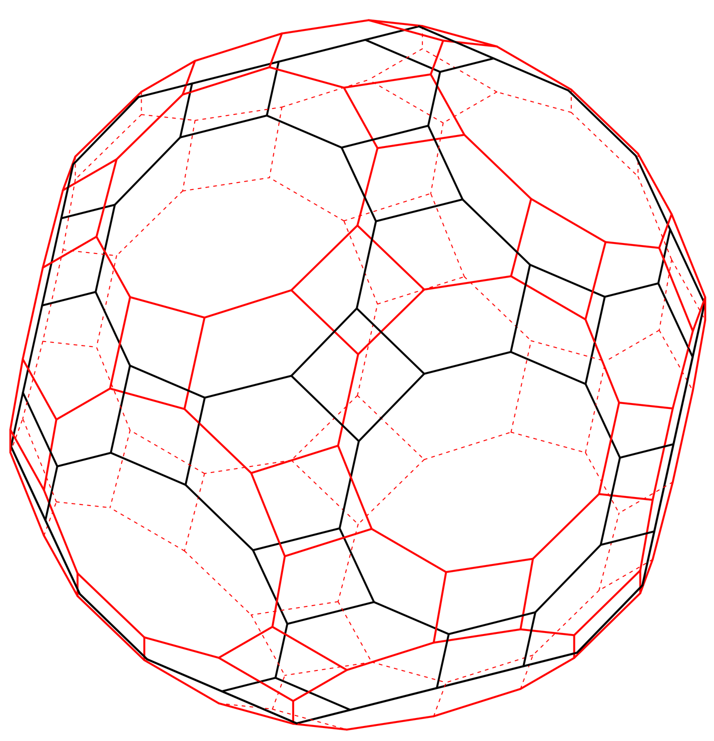
New and verified solutions of a Catalan solid, a Johnson solid, and an Archimedean solid [6]:



Deltoidal icositetrahedron



Elongated square gyrobicupola



Truncated icosidodecahedron

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