A hypergeometric proof that Iso is bijective

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Abstract

We provide a short and elementary proof of the main technical result of the recent article "*On the uniqueness of Clifford torus with prescribed isoperimetric ratio*" [4] by Thomas Yu and Jingmin Chen. The key of the new proof is an explicit expression of the central function (Iso, to be proved bijective) as a quotient of Gaussian hypergeometric functions.

In their recent paper [4], Thomas Yu and Jingmin Chen needed to prove, as a crucial intermediate result, that a certain real-valued function Iso (related to isoperimetric ratios of Clifford tori) is monotonic increasing. They reduced the proof of this fact to the positivity of a sequence of rational numbers $(d_n)_{n\geq 0}$, defined explicitly in terms of nested binomial sums. This positivity was subsequently proved by Stephen Melczer and Marc Mezzarobba [3], who used a computer-assisted approach relying on analytic combinatorics and rigorous numerics, combined with the fact (proved in [4]) that the sequence $(d_n)_{n\geq 0}$ satisfies an explicit linear recurrence of order seven with polynomial coefficients in n.

In this note, we provide an alternative, short and conceptual, proof of the monotonicity of the function Iso. Our approach is different in spirit from the ones in [4] and [3]. Our main result (Theorem 2 below) is that the function $\mathsf{Iso}(z)$ can be expressed in terms of Gaussian hypergeometric functions ${}_2F_1$ defined by

$$ {}_2F_1\left[\begin{matrix} a\ \ b \\ c \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1}$$

where $(a)_n$ denotes the rising factorial $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \in \mathbb{N}$.

In the notation of Yu and Chen, the function

$$\mathsf{Iso} : [0, \sqrt{2}-1) \to [3/2 \cdot (2\pi^2)^{-1/4}, 1)$$

is given as

$$\mathsf{Iso}(z) = 6\sqrt{\pi} \cdot \frac{V(z)}{A^{3/2}(z)}, \tag{2}$$

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where $A(z) = \sum_{n\geq 0} a_n z^{2n}$ and $V(z) = \sum_{n\geq 0} v_n z^{2n}$ are complex analytic functions in the disk $\{z : |z| < \sqrt{2} - 1\}$, given by the power series expansions

$$A(z) = \sqrt{2}\pi^2 \cdot \left(4 + 52z^2 + 477z^4 + 3809z^6 + \frac{451625}{16}z^8 + \cdots\right),$$

$$V(z) = \sqrt{2}\pi^2 \cdot \left(2 + 48z^2 + \frac{1269}{2}z^4 + 6600z^6 + \frac{1928025}{32}z^8 + \cdots\right).$$

The precise definitions of A and V are given in Section 4.3 of [4], notably in equations (4.2)–(4.3). Since the sequences $(a_n)_{\geq 0}$ and $(v_n)_{\geq 0}$ are expressed in terms of nested binomial sums, $A(z)$ and $V(z)$ satisfy linear differential equations with polynomial coefficients in z, that can be found and proved automatically using *creative telescoping* [2]. Yu and Chen, resp. Melczer and Mezzarobba, use this methodology to find a linear recurrence satisfied by the coefficients $(d_n)_{n\geq 0}$ of

$$F(z) := \frac{1}{4\pi^4}\cdot\left(\frac{2V'(\sqrt{z})A(\sqrt{z}) - 3V(\sqrt{z})A'(\sqrt{z})}{\sqrt{z}}\right) = 72+1932z+31248z^3+\cdots,$$

respectively a linear differential equation satisfied by the function $F(z)$.

Similarly, one can compute linear differential equations satisfied individually by

$$\bar{A}(z) := \frac{1}{\sqrt{2}\pi^2}\cdot A(\sqrt{z}) = 4+52\,z+477\,z^2+3809\,z^3+\frac{451625}{16}z^4+\frac{3195333}{16}z^5+\cdots$$

and by

$$\bar{V}(z) := \frac{1}{\sqrt{2}\pi^2}\cdot V(\sqrt{z}) = 2+48\,z+\frac{1269}{2}z^2+6600\,z^3+\frac{1928025}{32}z^4+\frac{2026101}{4}z^5+\cdots.$$

Concretely, $\bar{A}(z)$ and $\bar{V}(z)$ satisfy second-order linear differential equations:

$$z\,(z-1)\left(z^2-6\,z+1\right)(z+1)^2\,\bar{A}''+(z+1)\left(5\,z^4-8\,z^3-32\,z^2+28\,z-1\right)\bar{A}' + \left(4\,z^4+11\,z^3-z^2-43\,z+13\right)\bar{A}=0$$

and respectively

$$z\,(z-1)\,(z+1)\left(z^2-6\,z+1\right)^2\bar{V}'' + \left(z^2-6\,z+1\right)\left(7\,z^4-22\,z^3-18\,z^2+26\,z-1\right)\bar{V}' +3\,\left(3\,z^5-24\,z^4-2\,z^3+56\,z^2-25\,z+8\right)\bar{V}=0.$$

From these equations, we deduce the following closed-form expressions:

Theorem 1. *The following equalities hold for all $z \in \mathbb{R}$ with $0 \leq z \leq \sqrt{2}-1$:*

$$\bar{A}(z) = \frac{4\left(1-z^2\right)}{\left(z^2-6z+1\right)^2}\ \cdot\ {}_2F_1\left[\begin{matrix}-\frac{1}{2} & -\frac{1}{2}\\ & 1\end{matrix};\frac{4z}{(1-z)^2}\right]$$

and

$$\bar{V}(z) = \frac{2\left(1-z\right)^3}{\left(z^2-6z+1\right)^3}\ \cdot\ {}_2F_1\left[\begin{matrix}-\frac{3}{2} & -\frac{3}{2}\\ & 1\end{matrix};\frac{4z}{(1-z)^2}\right].$$

Proof. It is enough to check that the right-hand side expressions satisfy the same linear differential equations as $\bar{A}$ and $\bar{V}$, with the same initial conditions. $\square$

As a direct consequence of Theorem 1 and of definition (2) we get:

Theorem 2. *The function* Iso *admits the following closed-form expression:*

$$\mathsf{Iso}^2(z) = \frac{9\sqrt{2}}{8\pi} \cdot \frac{{}_2F_1\left[\begin{matrix}-\frac{3}{2}\ -\frac{3}{2}\\ 1\end{matrix}; \frac{4z^2}{(1-z^2)^2}\right]^2}{{}_2F_1\left[\begin{matrix}-\frac{1}{2}\ -\frac{1}{2}\\ 1\end{matrix}; \frac{4z^2}{(1-z^2)^2}\right]^3} \cdot \left(\frac{1-z^2}{1+z^2}\right)^3.$$

Using the expression in Theorem 2, we can now prove the main result of [4].

Theorem 3. Iso *is a monotonic increasing function and* $\lim_{z\to\sqrt{2}-1} \mathsf{Iso}(z) = 1$. *In particular,* Iso *is a bijection.*

Proof. The value of $\mathsf{Iso}^2(z)$ at $z = \sqrt{2}-1$ is equal to

$$\mathsf{Iso}^2(\sqrt{2}-1) = \frac{9\sqrt{2}}{8\pi} \cdot \frac{{}_2F_1\left[\begin{matrix}-\frac{3}{2}\ -\frac{3}{2}\\ 1\end{matrix}; 1\right]^2}{{}_2F_1\left[\begin{matrix}-\frac{1}{2}\ -\frac{1}{2}\\ 1\end{matrix}; 1\right]^3} \cdot \frac{\sqrt{2}}{4}.$$

From Gauss's summation theorem [1, Th. 2.2.2] it follows that ${}_2F_1\left[\begin{matrix}-\frac{3}{2}\ -\frac{3}{2}\\ 1\end{matrix}; 1\right] = 32/(3\pi)$ and ${}_2F_1\left[\begin{matrix}-\frac{1}{2}\ -\frac{1}{2}\\ 1\end{matrix}; 1\right] = 4/\pi$; therefore,

$$\mathsf{Iso}^2(\sqrt{2}-1) = \frac{9\sqrt{2}}{8\pi} \cdot \frac{(32/(3\pi))^2}{(4/\pi)^3} \cdot \frac{\sqrt{2}}{4} = 1.$$

It remains to prove that Iso is monotonic increasing. It is enough to show that

$$z \mapsto \frac{{}_2F_1\left[\begin{matrix}-\frac{3}{2}\ -\frac{3}{2}\\ 1\end{matrix}; \frac{4z}{(1-z)^2}\right]^2}{{}_2F_1\left[\begin{matrix}-\frac{1}{2}\ -\frac{1}{2}\\ 1\end{matrix}; \frac{4z}{(1-z)^2}\right]^3} \cdot \left(\frac{1-z}{1+z}\right)^3$$

is increasing on $[0, 3-2\sqrt{2})$. Equivalently, via the change of variables $x = \frac{4z}{(1-z)^2}$, it is enough to prove that the function

$$h \colon x \mapsto \frac{{}_2F_1\left[\begin{matrix}-\frac{3}{2}\ -\frac{3}{2}\\ 1\end{matrix}; x\right]^2}{{}_2F_1\left[\begin{matrix}-\frac{1}{2}\ -\frac{1}{2}\\ 1\end{matrix}; x\right]^3} \cdot (x+1)^{-\frac{3}{2}}$$

is increasing on $[0, 1)$. Clearly, h can be written as $h = f^3 \cdot g^2$, where

$$f(x) = \frac{\sqrt{x+1}}{{}_2F_1\left[\begin{matrix}-\frac{1}{2}\ -\frac{1}{2}\\ 1\end{matrix}; x\right]} \quad \text{and} \quad g(x) = \frac{{}_2F_1\left[\begin{matrix}-\frac{3}{2}\ -\frac{3}{2}\\ 1\end{matrix}; x\right]}{(x+1)^{\frac{3}{2}}}.$$

Hence, it is enough to prove that both f and g are increasing on $[0,1)$. We will actually prove a more general fact in Proposition 1, which may be of independent interest. Using that $w_{1/2} = 1/f$ and $w_{3/2} = g$, we deduce from Proposition 1 that both f and g are increasing. This concludes the proof of Theorem 3. $\square$

Proposition 1. *Let $a \geq 0$ and let $w_a : [0,1] \to \mathbb{R}$ be defined by*

$$w_a(x) = \frac{{}_2F_1\left[\begin{smallmatrix} -a & -a \\ & 1 \end{smallmatrix}; x\right]}{(x+1)^a}.$$

Then w_a is: decreasing if $0 < a < 1$; increasing if $a > 1$; constant if $a \in \{0,1\}$.

Proof. Clearly, if $a \in \{0,1\}$, then $w_a(x)$ is constant, equal to 1 on $[0,1]$.

Consider now the case $a > 0$ with $a \neq 1$. The derivative of $w_a(x)$ satisfies the hypergeometric identity

$$\frac{w_a'(x) \cdot (x+1)^{a+1}}{a \cdot (a-1) \cdot (1-x)^{2a}} = {}_2F_1\left[\begin{matrix} a+1 & a \\ & 2 \end{matrix}; x\right], \tag{3}$$

which is a direct consequence of Euler's transformation formula [1, Eq. (2.2.7), p. 68] and of Lemma 1 with a substituted by $-a$.

Since $a > 0$, the right-hand side of (3) has only positive Taylor coefficients, therefore it is positive on $[0,1)$. It follows that $w_a'(x) \geq 0$ on $[0,1]$ if $a - 1 > 0$, and $w_a'(x) \leq 0$ on $[0,1]$ if $a - 1 < 0$. Equivalently, w_a is increasing on $[0,1]$ if $a > 1$, and decreasing on $[0,1]$ if $a < 1$. $\square$

Lemma 1. *The following identity holds:*

$$(a+1)(1-x) \cdot {}_2F_1\left[\begin{matrix} a+1 & a+2 \\ & 2 \end{matrix}; x\right] = a(x+1) \cdot {}_2F_1\left[\begin{matrix} a+1 & a+1 \\ & 2 \end{matrix}; x\right] + {}_2F_1\left[\begin{matrix} a & a \\ & 1 \end{matrix}; x\right].$$

Proof. We will use two of the classical Gauss' contiguous relations [1, §2.5]:

$${}_2F_1\left[\begin{matrix} a+1 & b+1 \\ & c+1 \end{matrix}; x\right] = \frac{c}{bx} \cdot \left({}_2F_1\left[\begin{matrix} a+1 & b \\ & c \end{matrix}; x\right] - {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; x\right] \right) \tag{4}$$

and

$$a \cdot \left({}_2F_1\left[\begin{matrix} a+1 & b \\ & c \end{matrix}; x\right] - {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; x\right] \right) = \frac{(c-b) \cdot {}_2F_1\left[\begin{smallmatrix} a & b-1 \\ & c \end{smallmatrix}; x\right] + (b - c + ax) \cdot {}_2F_1\left[\begin{smallmatrix} a & b \\ & c \end{smallmatrix}; x\right]}{1-x}. \tag{5}$$

Applying (4) twice, once with $(b,c) = (a,1)$ and once with $(b,c) = (a+1,1)$, the proof of the lemma is reduced to that of the identity

$$(x-1) \cdot {}_2F_1\left[\begin{matrix} a+1 & a+1 \\ & 1 \end{matrix}; x\right] + 2 \cdot {}_2F_1\left[\begin{matrix} a & a+1 \\ & 1 \end{matrix}; x\right] = {}_2F_1\left[\begin{matrix} a & a \\ & 1 \end{matrix}; x\right],$$

which follows from (5) with $(b,c) = (a+1,1)$. $\square$

Remark 1. A natural question is whether the function Iso enjoys higher monotonicity properties. It can be easily seen that both Iso and its reciprocal $1/\mathsf{Iso}$ are neither convex nor concave. However we will prove that $z \mapsto \mathsf{Iso}(\sqrt{z})$ is concave and $z \mapsto 1/\mathsf{Iso}(\sqrt{z})$ is convex, on their domain of definition $[0, 3-2\sqrt{2})$.

First recall that $1/\mathsf{Iso}(\sqrt{z}) = \frac{2^{5/4}\cdot\sqrt{\pi}}{3} \cdot w_{1/2}(r(z))^{3/2} \cdot w_{3/2}(r(z))^{-1}$, where we set $r(z) = 4z/(1-z)^2$. Since $w_{1/2} = 1/f$ and $w_{3/2}^{-1} = 1/g$ are positive and decreasing, while r is nonnegative and increasing, proving that $w_{1/2} \circ r$ and $w_{3/2}^{-1} \circ r$ are both convex is enough to establish convexity of $z \mapsto 1/\mathsf{Iso}(\sqrt{z})$.

From (3) and the chain rule it follows that

$$\frac{\frac{\mathrm{d}}{\mathrm{d}z}\, w_a(r(z))}{4\,a\cdot(a-1)} = {}_2F_1\left[\begin{matrix} a+1 \;\; a \\ 2 \end{matrix}; \frac{4z}{(1-z)^2}\right] \cdot \frac{(1-6\,z+z^2)^{2a}}{(1-z)^{4a}} \cdot \frac{(1-z)^{2a-1}}{(1+z)^{2a+1}}. \qquad (6)$$

We can justify convexity of both $w_{1/2}(r(z))$ and $w_{3/2}(r(z))^{-1}$ if we can prove that the right-hand side of (6) is decreasing on $[0, 3-2\sqrt{2})$. Moreover, it is easy to see that $(1-z)^{2a-1}/(1+z)^{2a+1}$ is decreasing on this interval for $a > 3/2-\sqrt{2}$. Therefore, after changing variables $x = 4z/(1-z)^2$, it remains to show that

$${}_2F_1\left[\begin{matrix} a+1 \;\; a \\ 2 \end{matrix}; \frac{4z}{(1-z)^2}\right] \cdot \frac{(1-6\,z+z^2)^{2a}}{(1-z)^{4a}} = {}_2F_1\left[\begin{matrix} a+1 \;\; a \\ 2 \end{matrix}; x\right] \cdot (1-x)^{2a}$$

is decreasing for all $x \in [0,1)$. The derivative of the right-hand side is given by

$$-\left(\frac{a(3-a)}{2} \cdot {}_2F_1\left[\begin{matrix} a \;\; a+1 \\ 3 \end{matrix}; x\right] + \frac{a(a+1)x}{6} \cdot {}_2F_1\left[\begin{matrix} a+1 \;\; a+2 \\ 4 \end{matrix}; x\right]\right) \cdot (1-x)^{2a-1},$$

hence is indeed negative for all $x \in [0,1)$ if $0 < a < 3$. From this and (6) it follows that $1/\mathsf{Iso}(\sqrt{z}) = w_{1/2}(r(z))^{3/2} \cdot w_{3/2}(r(z))^{-1}$ is the product of two positive, decreasing and convex functions and therefore inherits these properties. Finally, this also shows that $\mathsf{Iso}(\sqrt{z})$ is both increasing and concave on $[0, 3-2\sqrt{2})$.

Remark 2. Bruno Salvy (private communication) found an alternative short proof of Proposition 1. The main idea is inspired by the Sturm–Liouville theory and the proof is based on the observation that $w_a(x)$ satisfies the linear differential equation (written in adjoint form):

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(\frac{1+x}{1-x}\right)^{2a} \cdot \frac{\mathrm{d}}{\mathrm{d}x} w_a(x)\right) = \frac{a(a-1)x}{(1+x)^2}\left(\frac{1+x}{1-x}\right)^{4a} \cdot w_a(x).$$

The right-hand side is positive on $[0,1)$ when $a > 1$ and negative if $0 < a < 1$. The same holds for its integral over $[0,t]$ for any $t < 1$. Looking at the left-hand side, this implies that $w_a' > 0$ whenever $a > 1$ and $w_a' < 0$ when $0 < a < 1$.

We note that the same idea allows for a different proof of Remark 1.

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